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Vector coherent state theory of the non-compact orthosymplectic superalgebras: II. Some selected examples

C Quesne†

Physique Nucléaire Théorique et Physique Mathématique CP229, Université Libre de Bruxelles, Bd du Triomphe, B1050 Bruxelles, Belgium

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Abstract. The techniques, presented in the first paper of the present series, for determining the conditions for the existence of star or grade star positive discrete series irreducible representations of $\mathrm{osp}(P/2N, \mathbb{R})$ ($P = 2M$ or $2M+1$), and the branching rule for their decomposition into a direct sum of $\mathrm{so}(P) \oplus \mathrm{sp}(2N, \mathbb{R})$ irreducible representations, as well as for constructing explicit matrix realizations, are illustrated with a few selected examples. The latter include the most general irreps of $\mathrm{osp}(1/2N, \mathbb{R})$, $\mathrm{osp}(2/2, \mathbb{R})$, $\mathrm{osp}(3/2, \mathbb{R})$, $\mathrm{osp}(4/2, \mathbb{R})$, $\mathrm{osp}(2/4, \mathbb{R})$, and the most degenerate irreps of $\mathrm{osp}(2/2N, \mathbb{R})$. In addition, all the information necessary for dealing with other cases amenable to a full analytic treatment is provided.

1. Introduction

The purpose of the present series of papers is to construct explicit matrix realizations for the positive discrete series irreps of the non-compact orthosymplectic superalgebras $\mathrm{osp}(P/2N, \mathbb{R})$, where $P = 2M$ or $2M+1$, in $\mathrm{osp}(P/2N, \mathbb{R}) \supset \mathrm{so}(P) \oplus \mathrm{sp}(2N, \mathbb{R}) \supset \mathrm{so}(P) \oplus \mathrm{u}(N)$ bases. Here we exploit the vector coherent state (VCS) and K-matrix general theory, expounded in the first paper of this series (henceforth referred to as I and whose equations will be quoted by their number preceded by I) (Quesne 1990c), to obtain detailed results for the star and grade star irreps of some low-dimensional superalgebras often encountered in physical applications.

Some of the results presented in this paper were already derived by other methods. The branching rule for the decomposition of the $\mathrm{osp}(1/2N, \mathbb{R})$ positive discrete series irreps into $\mathrm{sp}(2N, \mathbb{R})$ irreps and the matrix realization of the same in an $\mathrm{osp}(1/2N, \mathbb{R}) \supset \mathrm{sp}(2N, \mathbb{R}) \supset \mathrm{u}(N)$ basis were recently determined by a raising operator technique (Quesne 1989). A matrix realization of the $\mathrm{osp}(2/2, \mathbb{R})$ irreps in an $\mathrm{osp}(2/2, \mathbb{R}) \supset \mathrm{so}(2) \oplus \mathrm{sp}(2, \mathbb{R}) \supset \mathrm{so}(2) \oplus \mathrm{u}(1)$ basis was built by direct resolution of the supercommutation relations (Balantekin *et al* 1989). The branching rule for the decomposition of the $\mathrm{osp}(4/2, \mathbb{R})$ irreps into $\mathrm{so}(4) \oplus \mathrm{sp}(2, \mathbb{R})$ irreps was obtained by constructing the lowest-weight states of the latter in a super Fock space (Schmitt *et al* 1989).

Such results, however interesting they may be, always provide a partial solution to the problem in hand. The raising operator technique indeed proves rather tedious to extend to other superalgebras than $\mathrm{osp}(1/2N, \mathbb{R})$. The supercommutation relation direct resolution is restricted to very low-dimensional superalgebras. If the explicit construction of lowest-weight states enables the determination of branching rules, a lot of extra work is still needed before getting the corresponding matrix realizations.

† Directeur de recherches FNRS.

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On the contrary, the VCS and K-matrix combined theory exploits the full power of Wigner–Racah tensor calculus for the $\mathrm{so}(P)\oplus\mathrm{u}(N)$ subalgebra. It therefore provides a unified and systematic procedure for determining branching rules. Moreover, once the latter have been obtained, it is a rather simple matter to derive explicit matrix realizations. The only practical limitation to the applicability of the method lies in the necessity for a detailed knowledge of some $\mathrm{so}(P)$ and $\mathrm{u}(N)$ Racah coefficients.

This paper is organized as follows. In section 2, the branching rule and a matrix realization are determined for the $\mathrm{osp}(1/2N,\mathbb{R})$ star irreps. In sections 3 and 4, the same problems are considered for the $\mathrm{osp}(3/2N,\mathbb{R})$ and $\mathrm{osp}(4/2N,\mathbb{R})$ star irreps, respectively. After some general considerations valid for arbitrary N, the cases of $\mathrm{osp}(3/2,\mathbb{R})$ and $\mathrm{osp}(4/2,\mathbb{R})$ are solved in detail. In section 5, the $\mathrm{osp}(2/2N,\mathbb{R})$ star and grade star irreps are reviewed with special emphasis on the most degenerate irreps for arbitrary N and on the $\mathrm{osp}(2/4,\mathbb{R})$ general irreps. Finally, section 6 contains some concluding remarks.

2. The osp(1/2N, ℝ) superalgebras

The even part of the $\mathrm{osp}(1/2N,\mathbb{R})$ superalgebra is the $\mathrm{sp}(2N,\mathbb{R})$ algebra, generated by the operators $D^{\dagger}_{ij}$, D^{ij} and $E_i{}^j$, $i,j=1,\ldots,N$, where $E_i{}^j$, $i,j=1,\ldots,N$, span the stability subalgebra $\mathrm{u}(N)$. The odd part of $\mathrm{osp}(1/2N,\mathbb{R})$ has basis elements $K_i=\pm F^{\dagger}_i$ and F^i, $i=1,\ldots,N$ (Quesne 1989, 1990a). The $\mathrm{osp}(1/2N,\mathbb{R})$ positive discrete series irreps are characterized by $[\Omega\rangle=[\Omega_1\Omega_2\ldots\Omega_N\rangle$, where $\Omega_1,\Omega_2,\ldots,\Omega_N$ are integers satisfying the inequalities $\Omega_1\geqslant\Omega_2\geqslant\ldots\geqslant\Omega_N>N$. Their lowest-weight state belongs to the lowest-weight $\mathrm{u}(N)$ irrep $\{\Omega\}=\{\Omega_1\Omega_2\ldots\Omega_N\}$ contained in their carrier space. In the present section, we shall examine under which conditions these $\mathrm{osp}(1/2N,\mathbb{R})$ irreps are equivalent to star representations and we shall determine the branching rule for their decomposition into $\mathrm{sp}(2N,\mathbb{R})$ irreps, as well as the $\mathrm{u}(N)$ reduced matrix elements of the odd generators between two lowest-weight $\mathrm{u}(N)$ irrep basis states.

From (I3.1) and (I3.2), it follows that the $\mathrm{osp}(1/2N,\mathbb{R})$ VCS are parametrized by the complex variables $z_{ij}=z_{ji}$, $i,j=1,\ldots,N$, and the Grassmann variables θ_i, $i=1,\ldots,N$. The VCS space therefore consists of functions $\Psi(z,\boldsymbol{\theta})$, which are holomorphic in z_{ij}, polynomials in θ_i, and take vector values in the intrinsic subspace, i.e. in the $\mathrm{u}(N)$ irrep $\{\Omega\}$ carrier space. The VCS representation of $\mathrm{osp}(1/2N,\mathbb{R})$ depends on the intrinsic $\mathrm{u}(N)$ generators $\mathbb{E}_i{}^j$, as well as on the variables z_{ij}, θ_i, and the corresponding differential operators $\nabla^{ij}=(1+\delta_{ij})\partial/\partial z_{ij}$, $\partial^i=\partial/\partial\theta_i$. In Quesne (1990a), it is given in explicit form for the sign choice $K_i=F^{\dagger}_i$.

Both the odd generators K_i and the Grassmann variables θ_i are components of irreducible tensors $\mathfrak{K}$ and $\mathfrak{s}$, transforming under the $\mathrm{u}(N)$ irrep $\{1\dot{0}\}$. The Q polynomials considered in section 4 of I, are characterized by the $\mathrm{u}(N)$ irrep $\{\mu\}=\{1^l\dot{0}\}$, where l may run over the set $\{0,1,\ldots,N\}$, and by the row label γ, which may be chosen as the $\mathrm{u}(N)$ weight $(i_1i_2\ldots i_l)$ with $1\leqslant i_1<i_2<\ldots<i_l\leqslant N$. They are denoted by $Q^{\{1^l\dot{0}\}}_{(i_1\ldots i_l)}(\mathfrak{s})$ and their phase is fixed by choosing the highest-weight polynomial as

$$Q^{\{1^l\dot{0}\}}_{(12\ldots l)}(\mathfrak{s})=\theta_l\ldots\theta_2\theta_1. \tag{2.1}$$

According to (I4.21), orthonormal VBB basis states reducing the stability subalgebra $\mathrm{u}(N)$ are given by

$$|[\Omega\rangle\{1^l\dot{0}\}\langle\omega\rangle\{\nu\}\rho\{h\}\chi\rangle=[P^{\{\nu\}}(z)\times[Q^{\{1^l\dot{0}\}}(\mathfrak{s})\times|\{\Omega\}\rangle]^{\langle\omega\rangle}]^{\rho\{h\}}_{\chi} \tag{2.2}$$

where the polynomials P are defined in (I4.15), $|\{\Omega\}\alpha)$ denotes an intrinsic subspace basis state, and the labels $\omega_1, \omega_2, \ldots, \omega_N$ may take all the values satisfying the relations

$$\sum_{i=1}^{N} \omega_i = \sum_{i=1}^{N} \Omega_i + l \tag{2.3}$$

and

$$\Omega_1 \leqslant \omega_1 \leqslant \Omega_1 + 1 \qquad \Omega_i \leqslant \omega_i \leqslant \min(\Omega_i + 1, \omega_{i-1}) \qquad i = 2, \ldots, N. \tag{2.4}$$

Since the irrep $\{1^l\dot{0}\}$ is completely determined by $\{\omega\}$ through (2.3), it may be dropped and the u(N) VBB basis states written in shorthand notation as $|\langle\omega\rangle\{\nu\}\rho\{h\}\chi)$.

The K transformation maps the VBB basis states (2.2) onto VCS states classified by the following labels:

$$\begin{array}{ccccc} \mathrm{osp}(1/2N, \mathbb{R}) & \supset & \mathrm{sp}(2N, \mathbb{R}) & \supset & \mathrm{u}(N). \\ [\Omega\rangle & t = \{1^l\dot{0}\} & \langle\omega\rangle & \{\nu\}\rho & \{h\} \end{array} \tag{2.5}$$

All $\mathcal{K}(\{\omega\})$ submatrices are one dimensional.

As explained in I, to write the recursion relation satisfied by $\mathcal{K}\mathcal{K}^{\dagger}(\{\omega\})$ in explicit form, we need the values of the reduced matrix elements of $\mathfrak{s}$, defined in (I5.14) and (I5.24). Taking (2.1) into account, we obtain

$$(\{1^{l+1}\dot{0}\}\|\mathfrak{s}\|\{1^l\dot{0}\}) = \sqrt{l+1} \tag{2.6}$$

and

$$\begin{aligned} &(\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\mathfrak{s}\|\langle\omega\rangle\{\dot{0}\}\{\omega\}) \\ &\quad = (-1)^{l-m}\left(\prod_{k=1}^{l} (\Omega_{p_k} - \Omega_i + i - p_k + 1)(\Omega_{p_k} - \Omega_i + i - p_k)^{-1}\right)^{1/2} \end{aligned} \tag{2.7}$$

where

$$\{\omega\} = \{\Omega + \Delta^{(l)}(p_1 \ldots p_l)\} \qquad \{\omega'\} = \{\Omega + \Delta^{(l+1)}(p_1 \ldots p_m i p_{m+1} \ldots p_l)\} \tag{2.8}$$

and $\Delta^{(l)}(p_1 \ldots p_l)$ denotes a row vector of dimension N with vanishing entries everywhere except for the components $p_1, \ldots, p_l$, which have value unity. The u(N) Wigner and Racah coefficients required to evaluate (2.6) and (2.7) have been taken from Biedenharn and Louck (1968), and Le Blanc and Hecht (1987), respectively.

By taking (2.7) and the results of appendix 1 into account, it can be easily proved that the recursion relation (I5.11) for $\mathcal{K}\mathcal{K}^{\dagger}(\{\omega\})$ can be written as

$$\begin{aligned} \mathcal{K}\mathcal{K}^{\dagger}(\{\omega'\}) = \pm\frac{1}{2}\Bigg\{ & 2\Omega_i + l - 2i + 2 + \sum_s \Bigg[(\Omega_i - \Omega_{p_s} + p_s - i + 1)(\Omega_i + \Omega_{p_s} - p_s - i + 1)^{-1} \\ & \times\left(\prod_{k \neq s} (\Omega_{p_k} - \Omega_{p_s} + p_s - p_k + 1)(\Omega_{p_k} - \Omega_{p_s} + p_s - p_k)^{-1}\right)\Bigg]\Bigg\}\mathcal{K}\mathcal{K}^{\dagger}(\{\omega\}) \end{aligned} \tag{2.9}$$

where on the right-hand side s and k run over the range $1, \ldots, l$. The sum over s can be performed by using the complex function residue theory (Le Blanc and Rowe 1987, Quesne 1990b), so that the recursion relation finally takes the simpler form

$$\mathcal{K}\mathcal{K}^{\dagger}(\{\omega'\}) = \pm(\Omega_i - i + 1)\left(\prod_k (\Omega_i + \Omega_{p_k} - p_k - i + 2)(\Omega_i + \Omega_{p_k} - p_k - i + 1)^{-1}\right)\mathcal{K}\mathcal{K}^{\dagger}(\{\omega\}). \tag{2.10}$$

In (2.10), the lower sign choice can be immediately ruled out. It would indeed lead to the relation

$$\mathcal{K}\mathcal{K}^{+}(\{\Omega+\Delta^{(1)}(1)\}) = -\Omega_1 \mathcal{K}\mathcal{K}^{+}(\{\Omega\}) < 0 \tag{2.11}$$

$\mathcal{K}\mathcal{K}^{+}$ being normalized in such a way that $\mathcal{K}\mathcal{K}^{+}(\{\Omega\}) = 1$. Hence, the adjoint relation in sp($2N$, $\mathbb{R}$) can be extended to an adjoint relation in osp($1/2N$, $\mathbb{R}$) in a single way corresponding to $K_i = F_i^{+}$, if we impose that the irrep $[\Omega\rangle$ be equivalent to a star representation.

For the upper sign choice, the solution of (2.10) can be written as

$$\mathcal{K}\mathcal{K}^{+}(\{\omega\}) = \left(\prod_k (\Omega_{p_k} - p_k + 1)\right)\left(\prod_{k<q} (\Omega_{p_k} + \Omega_{p_q} - p_k - p_q + 2)(\Omega_{p_k} + \Omega_{p_q} - p_k - p_q + 1)^{-1}\right) \tag{2.12}$$

where $\{\omega\}$, defined in (2.8), satisfies the conditions (2.4). Since the right-hand side of (2.12) is always positive, all the VBB basis states $|\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi)$ are mapped onto (non-vanishing) VCS basis states. The branching rule for the decomposition of the star irreps $[\Omega\rangle$ into sp($2N$, $\mathbb{R}$) irreps $\langle\omega\rangle$ is therefore given by (see also Quesne 1989)

$$[\Omega\rangle \downarrow \sum_{\omega_1=\Omega_1}^{\Omega_1+1} \sum_{\omega_2=\Omega_2}^{\min(\Omega_2+1,\omega_1)} \cdots \sum_{\omega_N=\Omega_N}^{\min(\Omega_N+1,\omega_{N-1})} \oplus\langle\omega\rangle. \tag{2.13}$$

All such irreps are typical (Scheunert 1979).

Finally, by setting

$$\mathcal{K}(\{\omega\}) = \mathcal{K}^{+}(\{\omega\}) = [\mathcal{K}\mathcal{K}^{+}(\{\omega\})]^{1/2} \tag{2.14}$$

and taking (2.7) and (2.12) into account, the u(N) reduced matrix elements (I5.23) of the odd generators $\mathfrak{F}$ between two lowest-weight u(N) irrep basis states become

$$\begin{aligned}&(\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\gamma(\mathfrak{F})\|\langle\omega\rangle\{\dot{0}\}\{\omega\})\\ &\quad= (-1)^m\Bigg[(\Omega_i - i + 1)\Bigg(\prod_k (\Omega_i + \Omega_{p_k} - p_k - i + 2)(\Omega_i - \Omega_{p_k} + p_k - i - 1)\\ &\qquad\times[(\Omega_i + \Omega_{p_k} - p_k - i + 1)(\Omega_i - \Omega_{p_k} + p_k - i)]^{-1}\Bigg)\Bigg]^{1/2}\end{aligned} \tag{2.15}$$

where $\{\omega\}$ and $\{\omega'\}$ are defined in (2.8). By combining this relation with an appropriate u(N) Wigner coefficient (Biedenharn and Louck 1968), it is easy to get the matrix element of $\gamma(F_k^{+})$ between two sp($2N$, $\mathbb{R}$) irrep lowest-weight states. The result of such a calculation agrees with the formula obtained by means of a raising operator technique (Quesne 1989).

3. The osp($3/2N$, $\mathbb{R}$) superalgebras

3.1. General remarks

The even part of the osp($3/2N$, $\mathbb{R}$) superalgebra is the so(3)$\oplus$sp($2N$, $\mathbb{R}$) algebra, where sp($2N$, $\mathbb{R}$) is generated by D_{ij}^{+}, D^{ij}, $E_i^{\ j}$, $i, j = 1, \ldots, N$, and so(3) by B^{+}, B and C. Here we have dropped index a, which only takes the single value 1. It is convenient to renormalize B^{+} and B so as to obtain the operators

$$L_+ = \sqrt{2}B^{+} \qquad L_- = \sqrt{2}B \qquad L_0 = C \tag{3.1}$$

satisfying standard angular momentum commutation relations

$$[L_0, L_\pm] = \pm L_\pm \qquad [L_+, L_-] = 2L_0. \tag{3.2}$$

The odd generators are $I_i = \pm G_i^\dagger$, $K_i = \pm F_i^\dagger$, $H_i = \pm (J^i)^\dagger$, and G^i, F^i, J^i, $i = 1, \ldots, N$. The first (respectively, last) three are the components $\mathfrak{H}_{mi}$ (respectively, $\mathfrak{J}_{m\bar{i}}$) of an so(3)⊕u(N) irreducible tensor $\mathfrak{H}$ (respectively, $\mathfrak{J}$) transforming under the irrep [1]⊕$\{1\dot{0}\}$ (respectively [1]⊕$\{\dot{0}-1\}$). Here $m = 1, 0, -1$, i (respectively, $\bar{i}$) denotes the u(N) weight $(0\ldots010\ldots0)$ (respectively, $(0\ldots0-10\ldots0)$) with 1 (respectively, -1) in position i, and

$$\begin{aligned} &\mathfrak{H}_{1i} = I_i \qquad \mathfrak{H}_{0i} = K_i \qquad \mathfrak{H}_{-1i} = -H_i \\ &\mathfrak{J}_{1\bar{i}} = (-1)^i J^i \qquad \mathfrak{J}_{0\bar{i}} = (-1)^i F^i \qquad \mathfrak{J}_{-1\bar{i}} = (-1)^{i-1} G^i. \end{aligned} \tag{3.3}$$

The positive discrete series irreps of osp(3/2N, ℝ) are characterized by $[\Xi\Omega\rangle = [\Xi\Omega_1\ldots\Omega_N\rangle$, where $\Xi, \Omega_1, \ldots, \Omega_N$ are integers satisfying the inequalities $\Xi \geqslant 0$, and $\Omega_1 \geqslant \Omega_2 \geqslant \ldots \geqslant \Omega_N > N$. The corresponding VCS are parametrized by the complex variables $z_{ij} = z_{ji}$, $i, j = 1, \ldots, N$, and the Grassmann variables σ_i, θ_i, τ_i, $i = 1, \ldots, N$. The latter (and the corresponding differential operators) are the components $\mathfrak{s}_{mi}$ (respectively, $\mathfrak{d}_{m\bar{i}}$) of a [1]⊕$\{1\dot{0}\}$ (respectively, [1]⊕$\{\dot{0}-1\}$) irreducible tensor $\mathfrak{s}$ (respectively, $\mathfrak{d}$),

$$\begin{aligned} &\mathfrak{s}_{1i} = \sigma_i \qquad \mathfrak{s}_{0i} = \theta_i \qquad \mathfrak{s}_{-1i} = -\tau_i \\ &\mathfrak{d}_{1\bar{i}} = (-1)^i \partial/\partial\tau_i \qquad \mathfrak{d}_{0\bar{i}} = (-1)^i \partial/\partial\theta_i \qquad \mathfrak{d}_{-1\bar{i}} = (-1)^{i-1} \partial/\partial\sigma_i \end{aligned} \tag{3.4}$$

in accordance with (3.3). Comparison with (I6.8) and (I6.9) shows that the phase factor appearing in the latter is $u_3 = -1$. The VCS representation of osp(3/2N, ℝ) also depends on the intrinsic generators

$$\mathbb{L}_+ = \sqrt{2}\mathbb{B}^\dagger \qquad \mathbb{L}_- = \sqrt{2}\mathbb{B} \qquad \mathbb{L}_0 = \mathbb{C} \tag{3.5}$$

and $\mathbb{E}_i{}^j$, $i, j = 1, \ldots, N$.

The Q polynomials are characterized by an so(3) irrep $[\lambda]$, a u(N) irrep $\{\mu\} = \{\mu_1\mu_2\ldots\mu_N\}$, an additional label κ, and a row index γ. Here $3 \geqslant \mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_N \geqslant 0$, $[\lambda]$ may run over those so(3) irreps contained in the u(3) irrep $\{\tilde{\mu}\} = \{\tilde{\mu}_1\tilde{\mu}_2\tilde{\mu}_3\}$, κ distinguishes between repeated $[\lambda]$ irreps in $\{\tilde{\mu}\}$ and is needed only for $N \geqslant 4$ (Moshinsky *et al* 1975), and γ may be taken as $m(\mu)$, where $m = \lambda, \lambda - 1, \ldots, -\lambda$, and (μ) denotes a Gel'fand pattern of $\{\mu\}$ (Gel'fand and Tseitlin 1950). Whenever κ is not needed, the polynomials can be easily constructed from $Q^{[1]\{1\dot{0}\}}(\mathfrak{s}) = \mathfrak{s}$ by so(3)⊕u(N) couplings, followed by an appropriate normalization. The highest-weight second-degree polynomials, for instance, can be written as

$$Q^{[\lambda]\{\mu\}}_{\mathrm{hw}\,\mathrm{hw}}(\mathfrak{s}) = \frac{1}{\sqrt{2}}[Q^{[1]\{1\dot{0}\}}(\mathfrak{s}) \times Q^{[1]\{1\dot{0}\}}(\mathfrak{s})]^{[\lambda]\{\mu\}}_{\mathrm{hw}\,\mathrm{hw}} \tag{3.6}$$

or, in explicit form,

$$Q^{[1]\{2\dot{0}\}}_{\mathrm{hw}\,\mathrm{hw}}(\mathfrak{s}) = \theta_1\sigma_1 \tag{3.7}$$

and

$$Q^{[2]\{1^2\dot{0}\}}_{\mathrm{hw}\,\mathrm{hw}}(\mathfrak{s}) = \sigma_2\sigma_1 \qquad Q^{[0]\{1^2\dot{0}\}}{}_{\mathrm{hw}}(\mathfrak{s}) = 3^{-1/2}(\theta_1\theta_2 + \sigma_1\tau_2 + \tau_1\sigma_2) \qquad \text{if } N \geqslant 2. \tag{3.8}$$

Comparison of (3.8) with (I6.12) shows that the phase factor appearing in the latter is $w_3 = 1$.

By taking (3.7) into account and writing the components of the intrinsic irreducible tensor $\vec{\mathbb{T}}^{[1]\{\dot{0}\}}$ as

$$\mathbb{T}_m^{[1]\{\dot{0}\}} = \mathbb{L}_m \qquad m = 1, 0, -1 \tag{3.9}$$

where $\mathbb{L}_1 = -\mathbb{L}_+/\sqrt{2}$ and $\mathbb{L}_{-1} = \mathbb{L}_-/\sqrt{2}$, it can be easily proved that $\Gamma_1^{(0)}(\boldsymbol{D}^\dagger)$ takes the form (I6.10) with $v_3 = -\sqrt{6}$ or, equivalently,

$$\Gamma_1^{(0)}(\boldsymbol{D}^\dagger) = \sqrt{2} Q^{[1]\{2\dot{0}\}}(\mathfrak{s}) \cdot \vec{\mathbb{L}}. \tag{3.10}$$

The reduced matrix element of $\Gamma_1^{(0)}(\boldsymbol{D}^\dagger)$, appearing in the recursion relation (I5.11), can therefore be written as

$$\begin{aligned}(t'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\Gamma_1^{(0)}(\boldsymbol{D}^\dagger)\|t''[\xi']\langle\omega''\rangle\{\dot{0}\}\{\omega''\})\\ = -[2(2\lambda'+1)\Xi(\Xi+1)/(2\lambda''+1)]^{1/2} U(\Xi 1 \xi' \lambda'; \Xi \lambda'')\\ \times U(\{\Omega\}\{\mu''\}\{\omega'\}\{2\dot{0}\}; \{\omega''\}\zeta''\{\mu'\}\zeta')(\kappa'[\lambda']\{\mu'\}\|Q^{[1]\{2\dot{0}\}}(\mathfrak{s})\|\kappa''[\lambda'']\{\mu''\})\end{aligned} \tag{3.11}$$

where $t' = \kappa'[\lambda']\{\mu'\}\zeta'$, $t'' = \kappa''[\lambda'']\{\mu''\}\zeta''$, and, as a result of (3.6),

$$\begin{aligned}(\kappa'[\lambda']\{\mu'\}\|Q^{[1]\{2\dot{0}\}}(\mathfrak{s})\|\kappa''[\lambda'']\{\mu''\})\\ = \frac{1}{\sqrt{2}} \sum_{[\tilde{\lambda}]\{\tilde{\mu}\}} U(1 1 \lambda' \lambda''; 1 \tilde{\lambda}) U(\{1\dot{0}\}\{1\dot{0}\}\{\mu'\}\{\mu''\}; \{2\dot{0}\}\{\tilde{\mu}\})\\ \times \sum_{\tilde{\kappa}} (\kappa'[\lambda']\{\mu'\}\|\mathfrak{s}\|\tilde{\kappa}[\tilde{\lambda}]\{\tilde{\mu}\})(\tilde{\kappa}[\tilde{\lambda}]\{\tilde{\mu}\}\|\mathfrak{s}\|\kappa''[\lambda'']\{\mu''\}).\end{aligned} \tag{3.12}$$

On the right-hand side of (3.11) and (3.12), the first U coefficient is an so(3) Racah coefficient while the second one is a u(N) Racah coefficient.

It essentially remains to determine the reduced matrix elements of $\mathfrak{s}$ defined in (I5.14)†. Since they depend on the explicit form of the Q polynomials, they have to be calculated for each N value. In the next subsection, we detail the $N = 1$ case.

3.2. *The osp(3/2, ℝ) superalgebra*

In the osp(3/2, ℝ) case, the index i takes just the single value 1, and so may be dropped. As in (3.6), the Q polynomials can be constructed by successive so(3) couplings (since the u(1) couplings are trivial). Their highest-weight component can be written as

$$Q_0^{[0]\{0\}}(\mathfrak{s}) = 1 \qquad Q_1^{[1]\{1\}}(\mathfrak{s}) = \sigma \qquad Q_1^{[1]\{2\}}(\mathfrak{s}) = \theta\sigma \qquad Q_0^{[0]\{3\}}(\mathfrak{s}) = -\tau\theta\sigma. \tag{3.13}$$

The non-vanishing so(3) reduced matrix elements of $\mathfrak{s}$ between two Q polynomials, as defined in (I5.14), are then given by

$$([1]\{1\}\|\mathfrak{s}\|[0]\{0\}) = 1 \qquad ([1]\{2\}\|\mathfrak{s}\|[1]\{1\}) = \sqrt{2} \qquad ([0]\{3\}\|\mathfrak{s}\|[1]\{2\}) = \sqrt{3}. \tag{3.14}$$

Note that the phase of the Q polynomials has been chosen in such a way that all these reduced matrix elements are positive.

In shorthand notation, the orthonormal VBB basis states reducing the stability subalgebra so(3)⊕u(1) may be written as $|[\xi]\langle\omega\rangle\{h\}\chi\rangle$, since the u(1) irreps $\{\mu\}$ and $\{\nu\}$ are determined by $\{\omega\}$ and $\{h\}$ through the relations

$$\mu = \omega - \Omega \qquad \nu = h - \omega \tag{3.15}$$

† The so(3) phase $\psi([\xi])$, appearing in the reduced matrix element of $\mathfrak{b}$ given in (I5.25), is defined as $\psi([\xi]) = \xi$.

the so(3) irrep $[\lambda]$ is fixed by $\{\mu\}$, and hence by $\{\omega\}$, as shown in (3.13), and all couplings are multiplicity free. The allowed irreps $[\xi]$ and $\langle\omega\rangle$ are listed in columns 1 and 2 of table 1, and the conditions for their existence are displayed in column 5 of the same table. The latter result comes from the coupling rule of the angular momenta Ξ and λ to total angular momentum ξ.

Since their rows and columns are labelled by $t=[\lambda]\{\mu\}$, all $\mathcal{K}([\xi]\{\omega\})$ submatrices are one dimensional. The recursion relation (I5.11) satisfied by $\mathcal{K}\mathcal{K}^{\dagger}([\xi]\{\omega\})$ is easily written down because all reduced matrix elements only involve so(3) Racah coefficients. We obtain altogether thirteen equations, listed in appendix 2. Since the number of equations greatly exceeds the number of unknowns, of which there are only seven, the calculations can be easily cross-checked.

From the results of appendix 2, it is obvious that no positive semi-definite solution can be obtained for all $\mathcal{K}\mathcal{K}^{\dagger}([\xi]\{\omega\})$ submatrices when the lower sign is chosen in the adjoint relations for the odd generators. For the upper sign choice, we get the solution

$$\begin{aligned}
&\mathcal{K}\mathcal{K}^{\dagger}([\Xi+1]\{\Omega+1\})=\Omega-\Xi \qquad \mathcal{K}\mathcal{K}^{\dagger}([\Xi]\{\Omega+1\})=\Omega+1\\
&\mathcal{K}\mathcal{K}^{\dagger}([\Xi-1]\{\Omega+1\})=\Omega+\Xi+1 \qquad \mathcal{K}\mathcal{K}^{\dagger}([\Xi+1]\{\Omega+2\})=(\Omega+1)(\Omega-\Xi)\\
&\mathcal{K}\mathcal{K}^{\dagger}([\Xi]\{\Omega+2\})=\Omega^{-1}(\Omega+1)(\Omega-\Xi)(\Omega+\Xi+1)\\
&\mathcal{K}\mathcal{K}^{\dagger}([\Xi-1]\{\Omega+2\})=(\Omega+1)(\Omega+\Xi+1)\\
&\mathcal{K}\mathcal{K}^{\dagger}([\Xi]\{\Omega+3\})=(\Omega+2)(\Omega-\Xi)(\Omega+\Xi+1)
\end{aligned} \tag{3.16}$$

if and only if the condition

$$\Omega \geqslant \Xi \tag{3.17}$$

is satisfied. Whenever equation (3.17) is fulfilled, and only in such a case, the irrep $[\Xi\Omega\rangle$ is therefore equivalent to a star representation.

From (3.16), it is clear that if $\Omega>\Xi$, then all VBB basis states $|[\xi]\langle\omega\rangle\{\omega\}\chi)$ are mapped onto VCS basis states. On the contrary, as shown in column 6 of table 1, if $\Omega=\Xi$, then four VBB basis states are mapped onto the null vector whenever $\Xi\neq 0$. The branching rule for the decomposition of $[\Xi\Omega\rangle$ into so(3)⊕sp(2, ℝ) irreps $[\xi]\oplus\langle\omega\rangle$ results from combining the VBB and VCS conditions listed in columns 5 and 6 of table 1.

As a final point, it is an easy matter to obtain the so(3)⊕sp(2, ℝ) (triple) reduced matrix elements of the odd generators $\mathfrak{T}=(\mathfrak{H},\mathfrak{I})$ from (I5.22), (I5.23), (I5.35), (I5.36) and a relation similar to (2.14). They are listed in appendix 3 for the generic case $\Omega>\Xi>0$. These results also apply to the cases where $\Xi=0$ or $\Omega=\Xi$, provided only the allowed VCS basis states are retained.

Table 1. Branching rule for the decomposition of an osp(3/2, ℝ) star irrep $[\Xi\Omega\rangle$ into so(3)⊕sp(2, ℝ) irreps $[\xi]\oplus\langle\omega\rangle$.

$[\xi]$	$\langle\omega\rangle$	$[\lambda]$	$\{\mu\}$	VBB conditions	VCS conditions
$[\Xi]$	$\langle\Omega\rangle$	[0]	{0}	—	—
$[\Xi+1]$	$\langle\Omega+1\rangle$	[1]	{1}	—	$\Omega\neq\Xi$
$[\Xi]$	$\langle\Omega+1\rangle$	[1]	{1}	$\Xi\neq 0$	—
$[\Xi-1]$	$\langle\Omega+1\rangle$	[1]	{1}	$\Xi\neq 0$	—
$[\Xi+1]$	$\langle\Omega+2\rangle$	[1]	{2}	—	$\Omega\neq\Xi$
$[\Xi]$	$\langle\Omega+2\rangle$	[1]	{2}	$\Xi\neq 0$	$\Omega\neq\Xi$
$[\Xi-1]$	$\langle\Omega+2\rangle$	[1]	{2}	$\Xi\neq 0$	—
$[\Xi]$	$\langle\Omega+3\rangle$	[0]	{3}	—	$\Omega\neq\Xi$

4. The osp(4/2N, $\mathbb{R}$) superalgebras

4.1. General remarks

The even part of the osp(4/2N, $\mathbb{R}$) superalgebra is the so(4)$\oplus$sp(2N, $\mathbb{R}$) algebra, where sp(2N, $\mathbb{R}$) is generated by $D^\dagger_{ij}$, D^{ij}, $E_i{}^j$, $i, j = 1, \ldots, N$, and so(4) is spanned by $A^\dagger_{12}$, A^{12}, $C_1{}^1$, $C_1{}^2$, $C_2{}^1$, and $C_2{}^2$, or, alternatively, by the generators

$$S_{1+} = A^\dagger_{12} \qquad S_{1-} = A^{12} \qquad S_{10} = \tfrac{1}{2}(C_1{}^1 + C_2{}^2)$$
$$S_{2+} = C_1{}^2 \qquad S_{2-} = C_2{}^1 \qquad S_{20} = \tfrac{1}{2}(C_1{}^1 - C_2{}^2) \tag{4.1}$$

of the isomorphic su(2)$\oplus$su(2) algebra. Here each triple of operators S_{1+}, S_{1-}, S_{10}, and S_{2+}, S_{2-}, S_{20} satisfies commutation relations similar to (3.2), while any operator of the former set commutes with any operator of the latter. The odd raising generators $I_{ai} = \pm G^\dagger_{ai}$, $H_i^a = \pm(J_a{}^i)^\dagger$, $a = 1, 2$, $i = 1, \ldots, N$, are the components $\mathfrak{H}_{m_1 m_2 i}$ of a $[10]\oplus\{1\dot{0}\} \simeq (\tfrac{1}{2}, \tfrac{1}{2})\oplus\{1\dot{0}\}$ irreducible tensor $\mathfrak{H}$ with respect to so(4)$\oplus$u(N) $\simeq$ [su(2)$\oplus$su(2)]$\oplus$u(N), where m_1, $m_2 = \tfrac{1}{2}, -\tfrac{1}{2}$, and $i = 1, \ldots, N$:

$$\mathfrak{H}_{1/2\ 1/2\ i} = I_{1i} \qquad \mathfrak{H}_{1/2\ -1/2\ i} = I_{2i}$$
$$\mathfrak{H}_{-1/2\ 1/2\ i} = H_i^2 \qquad \mathfrak{H}_{-1/2\ -1/2\ i} = -H_i^1. \tag{4.2}$$

In the same way, the odd lowering generators G^{ai}, $J_a{}^i$, $a = 1, 2$, $i = 1, \ldots, N$, are the components $\mathfrak{I}_{m_1 m_2 \bar{i}}$ of a $[10]\oplus\{\dot{0}-1\} \simeq (\tfrac{1}{2}, \tfrac{1}{2})\oplus\{\dot{0}-1\}$ irreducible tensor $\mathfrak{I}$, where m_1, $m_2 = \tfrac{1}{2}, -\tfrac{1}{2}$, and $i = 1, \ldots, N$:

$$\mathfrak{I}_{1/2\ 1/2\ \bar{i}} = (-1)^i J_1{}^i \qquad \mathfrak{I}_{1/2\ -1/2\ \bar{i}} = (-1)^i J_2{}^i$$
$$\mathfrak{I}_{-1/2\ 1/2\ \bar{i}} = (-1)^i G^{2i} \qquad \mathfrak{I}_{-1/2\ -1/2\ \bar{i}} = (-1)^{i-1} G^{1i}. \tag{4.3}$$

The positive discrete series irreps of osp(4/2N, $\mathbb{R}$) are characterized by $[\Xi\Omega\rangle = [\Xi_1\Xi_2\Omega_1 \ldots \Omega_N\rangle$, where $\Xi_1, \Xi_2, \Omega_1, \ldots, \Omega_N$ are integers satisfying the inequalities $\Xi_1 \geqslant |\Xi_2|$, and $\Omega_1 \geqslant \Omega_2 \geqslant \ldots \geqslant \Omega_N > N$. The lowest-weight so(4) irrep $[\Xi_1\Xi_2]$ can also be denoted by (S_1, S_2), where $S_1 = \tfrac{1}{2}(\Xi_1 + \Xi_2)$ and $S_2 = \tfrac{1}{2}(\Xi_1 - \Xi_2)$ specify the irreps of the isomorphic su(2)$\oplus$su(2) algebra and are simultaneously integer or half integer.

The corresponding VCS are now parametrized by the complex variables $z_{ij} = z_{ji}$, $i, j = 1, \ldots, N$, and the Grassmann variables σ_{ai}, τ_i^a, $a = 1, 2$, $i = 1, \ldots, N$. In accordance with (4.2) and (4.3), the latter (and the corresponding differential operators) are the components $\mathfrak{s}_{m_1 m_2 i}$ (respectively, $\mathfrak{d}_{m_1 m_2 \bar{i}}$) of a $[10]\oplus\{1\dot{0}\} \simeq (\tfrac{1}{2}, \tfrac{1}{2})\oplus\{1\dot{0}\}$ (respectively, $[10]\oplus\{\dot{0}-1\} \simeq (\tfrac{1}{2}, \tfrac{1}{2})\oplus\{\dot{0}-1\}$) irreducible tensor $\mathfrak{s}$ (respectively, $\mathfrak{d}$):

$$\mathfrak{s}_{1/2\ 1/2\ i} = \sigma_{1i} \qquad \mathfrak{s}_{1/2\ -1/2\ i} = \sigma_{2i}$$
$$\mathfrak{s}_{-1/2\ 1/2\ i} = \tau_i^2 \qquad \mathfrak{s}_{-1/2\ -1/2\ i} = -\tau_i^1$$
$$\mathfrak{d}_{1/2\ 1/2\ \bar{i}} = (-1)^i \partial_1{}^i \qquad \mathfrak{d}_{1/2\ -1/2\ \bar{i}} = (-1)^i \partial_2{}^i$$
$$\mathfrak{d}_{-1/2\ 1/2\ \bar{i}} = (-1)^i \partial^{2i} \qquad \mathfrak{d}_{-1/2\ -1/2\ \bar{i}} = (-1)^{i-1} \partial^{1i}. \tag{4.4}$$

Comparison with (I6.8) and (I6.9) shows that the phase factor appearing in the latter is $u_4 = -1$. The VCS representation of osp(4/2N, $\mathbb{R}$) also makes use of the intrinsic generators $\mathbb{E}_i{}^j$, $i, j = 1, \ldots, N$, and $\mathbb{S}_{1+}$, $\mathbb{S}_{1-}$, $\mathbb{S}_{10}$, $\mathbb{S}_{2+}$, $\mathbb{S}_{2-}$, $\mathbb{S}_{20}$, the latter being defined in terms of $\mathbb{A}^\dagger_{12}$, $\mathbb{A}^{12}$, $\mathbb{C}_1{}^1$, $\mathbb{C}_1{}^2$, $\mathbb{C}_2{}^1$, and $\mathbb{C}_2{}^2$ by relations analogous to (4.1).

The Q polynomials are characterized by an so(4) $\simeq$ su(2)$\oplus$su(2) irrep $[\lambda] = [\lambda_1\lambda_2] \simeq (k_1, k_2)$, a u($N$) irrep $\{\mu\} = \{\mu_1\mu_2 \ldots \mu_N\}$, a set of two additional labels κ, and a row index γ. Here $4 \geqslant \mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_N \geqslant 0$, $[\lambda]$ may run over those so(4) irreps contained in the u(4) irrep $\{\tilde{\mu}\}=\{\tilde{\mu}_1\tilde{\mu}_2\tilde{\mu}_3\tilde{\mu}_4\}$, k_1 and k_2 are given in terms of λ_1 and λ_2 by $k_1=\frac{1}{2}(\lambda_1+\lambda_2)$, $k_2=\frac{1}{2}(\lambda_1-\lambda_2)$, κ distinguishes between repeated $[\lambda]$ irreps in $\{\tilde{\mu}\}$ and is needed only for $N\geqslant 3$ (Quesne 1976, 1977), and γ may be taken as $m_1m_2(\mu)$, where $m_1=k_1, k_1-1,\ldots,-k_1$, $m_2=k_2, k_2-1,\ldots,-k_2$, and (μ) denotes a Gel'fand pattern of $\{\mu\}$.

Whenever κ is not needed, the polynomials can be constructed by su(2)⊕su(2)⊕u(N) couplings, followed by an appropriate normalization, in a way similar to that shown for osp(3/2N, ℝ) in (3.6). The highest-weight second-degree polynomials are given by

$$\begin{aligned} Q^{[11]\{2\dot{0}\}}_{\mathrm{hw}\ \mathrm{hw}}(\mathfrak{s}) &= Q^{(1,0)\{2\dot{0}\}}_{\mathrm{hw}\ \mathrm{hw}}(\mathfrak{s}) = \sigma_{21}\sigma_{11} \\ Q^{[1-1]\{2\dot{0}\}}_{\mathrm{hw}\ \mathrm{hw}}(\mathfrak{s}) &= Q^{(0,1)\{2\dot{0}\}}_{\mathrm{hw}\ \mathrm{hw}}(\mathfrak{s}) = {\tau_1}^2\sigma_{11} \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} Q^{[20]\{1^2\dot{0}\}}_{\mathrm{hw}\ \mathrm{hw}}(\mathfrak{s}) &= Q^{(1,1)\{1^2\dot{0}\}}_{\mathrm{hw}\ \mathrm{hw}}(\mathfrak{s}) = \sigma_{12}\sigma_{11} \\ Q^{[00]\{1^2\dot{0}\}}_{\ \mathrm{hw}}(\mathfrak{s}) &= Q^{(0,0)\{1^2\dot{0}\}}_{\ \mathrm{hw}}(\mathfrak{s}) = \tfrac{1}{2}(\sigma_{11}{\tau_2}^1+\sigma_{21}{\tau_2}^2+{\tau_1}^1\sigma_{12}+{\tau_1}^2\sigma_{22}) \qquad \text{if } N>1. \end{aligned} \tag{4.6}$$

Comparison of (4.6) with (I6.12) shows that the phase factor appearing in the latter is $w_4=1$.

By taking (4.5) into account and noting that the components of the intrinsic irreducible tensors $\mathbb{T}^{[11]\{\dot{0}\}}\simeq\mathbb{T}^{(1,0)\{\dot{0}\}}$ and $\mathbb{T}^{[1-1]\{\dot{0}\}}\simeq\mathbb{T}^{(0,1)\{\dot{0}\}}$ are

$$\mathbb{T}^{(1,0)\{\dot{0}\}}_{m0}=\mathbb{S}_{1m} \qquad \mathbb{T}^{(0,1)\{\dot{0}\}}_{0m}=\mathbb{S}_{2m} \qquad m=1,0,-1 \tag{4.7}$$

it can be easily proved that $\Gamma_1^{(0)}(\boldsymbol{D}^+)$ takes the form (I6.10) with $v_4=-2\sqrt{3}$ or, equivalently,

$$\Gamma_1^{(0)}(\boldsymbol{D}^+)=2[Q^{(1,0)\{2\dot{0}\}}(\mathfrak{s})\cdot\vec{\mathbb{S}}_1+Q^{(0,1)\{2\dot{0}\}}(\mathfrak{s})\cdot\vec{\mathbb{S}}_2]. \tag{4.8}$$

The reduced matrix element of $\Gamma_1^{(0)}(\boldsymbol{D}^+)$, appearing in the recursion relation (I5.11), can therefore be written as

$$\begin{aligned} &(t'[\xi']\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\Gamma_1^{(0)}(\boldsymbol{D}^+)\|t''[\xi']\langle\omega''\rangle\{\dot{0}\}\{\omega''\}) \\ &\quad=-2U(\{\Omega\}\{\mu''\}\{\omega'\}\{2\dot{0}\};\{\omega''\}\zeta''\{\mu'\}\zeta')\{[(2k_1'+1)S_1(S_1+1)/(2k_1''+1)]^{1/2} \\ &\quad\times U(S_11s_1'k_1';S_1k_1'')(\kappa'(k_1',k_2')\{\mu'\}\|Q^{(1,0)\{2\dot{0}\}}(\mathfrak{s})\|\kappa''(k_1'',k_2'')\{\mu''\}) \\ &\quad+[(2k_2'+1)S_2(S_2+1)/(2k_2''+1)]^{1/2}U(S_21s_2'k_2';S_2k_2'') \\ &\quad\times(\kappa'(k_1',k_2')\{\mu'\}\|Q^{(0,1)\{2\dot{0}\}}(\mathfrak{s})\|\kappa''(k_1'',k_2'')\{\mu''\})\} \end{aligned} \tag{4.9}$$

where $t'=\kappa'[\lambda']\{\mu'\}\zeta'$, $t''=\kappa''[\lambda'']\{\mu''\}\zeta''$, $[\lambda']\simeq(k_1',k_2')$, $[\lambda'']\simeq(k_1'',k_2'')$, $[\xi']\simeq(s_1',s_2')$, and

$$\begin{aligned} &(\kappa'(k_1',k_2')\{\mu'\}\|Q^{(1,0)\{2\dot{0}\}}(\mathfrak{s})\|\kappa''(k_1'',k_2'')\{\mu''\}) \\ &\quad=\frac{1}{\sqrt{2}}\delta_{k_2',k_2''}\sum_{\tilde{k}_1\tilde{k}_2\{\tilde{\mu}\}}U(\tfrac{1}{2}\tfrac{1}{2}k_1'k_1'';1\tilde{k}_1)U(\tfrac{1}{2}\tfrac{1}{2}k_2'k_2';0\tilde{k}_2) \\ &\quad\times U(\{1\dot{0}\}\{1\dot{0}\}\{\mu'\}\{\mu''\};\{2\dot{0}\}\{\tilde{\mu}\}) \\ &\quad\times\sum_{\tilde{\kappa}}(\kappa'(k_1',k_2')\{\mu'\}\|\mathfrak{s}\|\tilde{\kappa}(\tilde{k}_1,\tilde{k}_2)\{\tilde{\mu}\}) \\ &\quad\times(\tilde{\kappa}(\tilde{k}_1,\tilde{k}_2)\{\tilde{\mu}\}\|\mathfrak{s}\|\kappa''(k_1'',k_2'')\{\mu''\}) \end{aligned} \tag{4.10}$$

and the reduced matrix element of $Q^{(0,1)\{2\dot{0}\}}(\mathfrak{s})$ is given by a relation similar to (4.10). The right-hand sides of (4.9) and (4.10) contain a u(N) Racah coefficient in addition to some standard su(2) Racah coefficients.

As in the osp$(3/2N, \mathbb{R})$ case, it remains to determine the reduced matrix elements of $\mathfrak{s}^\dagger$. In the next subsection, we detail the $N=1$ example.

4.2. The osp$(4/2, \mathbb{R})$ superalgebra

In (4.2)-(4.4), index i now takes the single value 1 and may therefore be dropped. The Q polynomials can be constructed by su(2)$\oplus$su(2) couplings and subsequent normalization. Their highest-weight component can be written as

$$
\begin{aligned}
&Q^{[00]\{0\}}(\mathfrak{s}) = Q_{00}^{(0,0)\{0\}}(\mathfrak{s}) = 1 \qquad && Q_{\rm hw}^{[10]\{1\}}(\mathfrak{s}) = Q_{1/2\ 1/2}^{(1/2,1/2)\{1\}}(\mathfrak{s}) = \sigma_1 \\
&Q_{\rm hw}^{[11]\{2\}}(\mathfrak{s}) = Q_{10}^{(1,0)\{2\}}(\mathfrak{s}) = \sigma_2\sigma_1 \qquad && Q_{\rm hw}^{[1-1]\{2\}}(\mathfrak{s}) = Q_{01}^{(0,1)\{2\}}(\mathfrak{s}) = \tau^2\sigma_1 \\
&Q_{\rm hw}^{[10]\{3\}}(\mathfrak{s}) = Q_{1/2\ 1/2}^{(1/2,1/2)\{3\}}(\mathfrak{s}) = \tau^2\sigma_2\sigma_1 \qquad && Q^{[00]\{4\}}(\mathfrak{s}) = Q_{00}^{(0,0)\{4\}}(\mathfrak{s}) = \tau^2\tau^1\sigma_2\sigma_1.
\end{aligned}
\tag{4.11}
$$

The non-vanishing su(2)$\oplus$su(2) reduced matrix elements of $\mathfrak{s}$ between two Q polynomials are then given by

$$
\begin{aligned}
&((\tfrac{1}{2},\tfrac{1}{2})\{1\}\|\mathfrak{s}\|(0,0)\{0\}) = 1 \\
&((1,0)\{2\}\|\mathfrak{s}\|(\tfrac{1}{2},\tfrac{1}{2})\{1\}) = ((0,1)\{2\}\|\mathfrak{s}\|(\tfrac{1}{2},\tfrac{1}{2})\{1\}) = \sqrt{2} \\
&((\tfrac{1}{2},\tfrac{1}{2})\{3\}\|\mathfrak{s}\|(1,0)\{2\}) = -((\tfrac{1}{2},\tfrac{1}{2})\{3\}\|\mathfrak{s}\|(0,1)\{2\}) = \sqrt{\tfrac{3}{2}} \\
&((0,0)\{4\}\|\mathfrak{s}\|(\tfrac{1}{2},\tfrac{1}{2})\{3\}) = 2.
\end{aligned}
\tag{4.12}
$$

As a matter of fact, the phase of the Q polynomials has been fixed so that the non-vanishing reduced matrix elements $((k_1', k_2')\{\mu+1\}\|\mathfrak{s}\|(k_1, k_2)\{\mu\})$ will be positive whenever (k_1, k_2) is the highest-weight su(2)$\oplus$su(2) irrep corresponding to $\{\mu\}$.

In shorthand notation, the orthonormal VBB basis states reducing the stability subalgebra so(4)$\oplus$u(1) may be written as $|[\lambda_1\lambda_2][\xi_1\xi_2]\langle\omega\rangle\{h\}\chi\rangle$, since the u(1) irreps $\{\mu\}$ and $\{\nu\}$ are determined by $\{\omega\}$ and $\{h\}$ through (3.15) and all couplings are multiplicity free. Contrary to what happens in the osp$(3/2, \mathbb{R})$ case, the so(4) irrep $[\lambda_1\lambda_2]$ is not fixed by $\{\mu\}$, as equation (4.11) shows that both $[11] \simeq (1,0)$ and $[1\ -1] \simeq (0,1)$ are associated with $\{2\}$. The allowed irreps $[\xi_1\xi_2] \simeq (s_1, s_2)$ and $\langle\omega\rangle$ are listed in columns 1 and 2 of table 2, and the conditions for their existence are displayed in column 5 of the same table. The latter result from the coupling rules of the angular momenta S_1 and k_1, S_2 and k_2 to the total angular momenta s_1 and s_2, respectively.

Since their rows and columns are labelled by $t = [\lambda_1\lambda_2]\{\mu\}$, all $\mathcal{K}([\xi_1\xi_2]\{\omega\})$ submatrices are one-dimensional, except for $\mathcal{K}([\Xi_1\Xi_2]\{\Omega+2\})$ which is two-dimensional for $\Xi_1 \neq |\Xi_2|$. In the latter case, we shall abbreviate t by $t = [11]$ or $[1\ -1]$.

The recursion relation (I5.11) satisfied by $\mathcal{K}\mathcal{K}^\dagger([\xi_1\xi_2]\{\omega\})$ only depends on some su(2) Racah coefficients and is therefore easily written down. There are altogether forty equations, which have to be satisfied by sixteen unknowns. Since there is again no positive semi-definite solution for the lower sign choice in the adjoint relations for

† The so(4) phase appearing in (I5.25) is defined as $\psi([\xi]) = \psi(s_1, s_2) = s_1 + s_2 = \xi_1$.

Table 2. Branching rule for the decomposition of an osp(4/2, ℝ) star irrep $[\Xi_1\Xi_2\Omega\rangle$ into so(4)⊕sp(2, ℝ) irreps $[\xi_1\xi_2]\oplus\langle\omega\rangle$.

$[\xi_1\xi_2]$	$\langle\omega\rangle$	$[\lambda_1\lambda_2]$	$\{\mu\}$	VBB conditions	VCS conditions
$[\Xi_1\Xi_2]$	$\langle\Omega\rangle$	[00]	{0}	—	—
$[\Xi_1+1\Xi_2]$	$\langle\Omega+1\rangle$	[10]	{1}	—	$\Omega\neq\Xi_1$
$[\Xi_1\Xi_2+1]$	$\langle\Omega+1\rangle$	[10]	{1}	$\Xi_1\neq\Xi_2$	—
$[\Xi_1\Xi_2-1]$	$\langle\Omega+1\rangle$	[10]	{1}	$\Xi_1\neq-\Xi_2$	—
$[\Xi_1-1\Xi_2]$	$\langle\Omega+1\rangle$	[10]	{1}	$\Xi_1\neq\Xi_2, -\Xi_2$	—
$[\Xi_1+1\Xi_2+1]$	$\langle\Omega+2\rangle$	[11]	{2}	—	$\Omega\neq\Xi_1$
$[\Xi_1+1\Xi_2-1]$	$\langle\Omega+2\rangle$	[1 −1]	{2}	—	$\Omega\neq\Xi_1$
$[\Xi_1\Xi_2]$	$\langle\Omega+2\rangle$	[11]	{2}	$\Xi_1\neq-\Xi_2$	—†
		[1 −1]	{2}	$\Xi_1\neq\Xi_2$	$\Omega\neq\Xi_1$‡
$[\Xi_1-1\Xi_2+1]$	$\langle\Omega+2\rangle$	[1 −1]	{2}	$\Xi_1\neq\Xi_2, \Xi_2+1$	—
$[\Xi_1-1\Xi_2-1]$	$\langle\Omega+2\rangle$	[11]	{2}	$\Xi_1\neq-\Xi_2, -\Xi_2+1$	—
$[\Xi_1+1\Xi_2]$	$\langle\Omega+3\rangle$	[10]	{3}	—	$\Omega\neq\Xi_1$
$[\Xi_1\Xi_2+1]$	$\langle\Omega+3\rangle$	[10]	{3}	$\Xi_1\neq\Xi_2$	$\Omega\neq\Xi_1$
$[\Xi_1\Xi_2-1]$	$\langle\Omega+3\rangle$	[10]	{3}	$\Xi_1\neq-\Xi_2$	$\Omega\neq\Xi_1$
$[\Xi_1-1\Xi_2]$	$\langle\Omega+3\rangle$	[10]	{3}	$\Xi_1\neq\Xi_2, -\Xi_2$	—
$[\Xi_1\Xi_2]$	$\langle\Omega+4\rangle$	[00]	{4}	—	$\Omega\neq\Xi_1$

† Condition valid for eigenvalue d_1. The latter does not exist whenever $\Xi_1=\Xi_2, -\Xi_2$.
‡ Condition valid for eigenvalue d_2.

the odd generators, we shall restrict here to the upper sign choice. In such a case, we obtain the solution

$$\mathcal{K}\mathcal{K}^\dagger([\Xi_1\pm1\Xi_2]\{\Omega+1\})=\Omega+1\mp(\Xi_1+1) \qquad \mathcal{K}\mathcal{K}^\dagger([\Xi_1\Xi_2\pm1]\{\Omega+1\})=\Omega\mp\Xi_2+1$$

$$\mathcal{K}\mathcal{K}^\dagger([\Xi_1+1\Xi_2\pm1]\{\Omega+2\})=(\Omega-\Xi_1)(\Omega\mp\Xi_2+1)$$

$$\mathcal{K}\mathcal{K}^\dagger([\Xi_1-1\Xi_2\pm1]\{\Omega+2\})=(\Omega+\Xi_1+2)(\Omega\mp\Xi_2+1)$$

$$(\mathcal{K}\mathcal{K}^\dagger([\Xi_1\Xi_2]\{\Omega+2\}))_{[1\ \pm1][1\ \pm1]}$$
$$=(2\Omega)^{-1}[2\Omega(\Omega+1)(\Omega+2)-(\Omega+1)C_1(\Xi_1,\Xi_2)\mp2C_2(\Xi_1,\Xi_2)] \tag{4.13}$$

$$(\mathcal{K}\mathcal{K}^\dagger([\Xi_1\Xi_2]\{\Omega+2\}))_{[11][1\ -1]}=-(2\Omega)^{-1}(\Omega+1)[C_1^2(\Xi_1,\Xi_2)-4C_2^2(\Xi_1,\Xi_2)]^{1/2}$$

$$\mathcal{K}\mathcal{K}^\dagger([\Xi_1\pm1\Xi_2]\{\Omega+3\})=(\Omega+1)^{-1}(\Omega+2)[\Omega+1\mp(\Xi_1+1)](\Omega-\Xi_2+1)(\Omega+\Xi_2+1)$$

$$\mathcal{K}\mathcal{K}^\dagger([\Xi_1\Xi_2\pm1]\{\Omega+3\})=(\Omega+1)^{-1}(\Omega+2)(\Omega-\Xi_1)(\Omega+\Xi_1+2)(\Omega\mp\Xi_2+1)$$

$$\mathcal{K}\mathcal{K}^\dagger([\Xi_1\Xi_2]\{\Omega+4\})=(\Omega+1)^{-1}(\Omega+3)(\Omega-\Xi_1)(\Omega+\Xi_1+2)(\Omega-\Xi_2+1)(\Omega+\Xi_2+1)$$

if and only if the condition

$$\Omega\geqslant\Xi_1 \tag{4.14}$$

is satisfied. In (4.13),

$$C_1(\Xi_1,\Xi_2)=\Xi_1(\Xi_1+2)+\Xi_2^2 \qquad C_2(\Xi_1,\Xi_2)=(\Xi_1+1)\Xi_2 \tag{4.15}$$

are the eigenvalues of the two so(4) Casimir operators corresponding to the irrep $[\Xi_1\Xi_2]$. To check that the 2×2 matrix $\mathcal{K}\mathcal{K}^\dagger([\Xi_1\Xi_2]\{\Omega+2\})$ is positive semi-definite whenever (4.14) is fulfilled, it is sufficient to show that its trace and its determinant are not negative. From (4.13) and (4.14), it follows that

$$\begin{aligned} \operatorname{tr} \mathcal{K}\mathcal{K}^{\dagger}([\Xi_1\Xi_2]\{\Omega+2\}) &= \Omega^{-1}(\Omega+1)[2(\Omega+1)^2-(\Xi_1+1)^2-\Xi_2{}^2-1] \geqslant 0 \\ \det \mathcal{K}\mathcal{K}^{\dagger}([\Xi_1\Xi_2]\{\Omega+2\}) &= \Omega^{-1}(\Omega+2)[(\Omega+1)^2-(\Xi_1+1)^2][(\Omega+1)^2-\Xi_2{}^2] \geqslant 0. \end{aligned} \tag{4.16}$$

Whenever condition (4.14) is satisfied, and only in such a case, the irrep $[\Xi_1\Xi_2\Omega\rangle$ is therefore equivalent to a star representation.

If $\Omega > \Xi_1$, then all the VBB basis states $|[\lambda_1\lambda_2][\xi_1\xi_2]\langle\omega\rangle\{\omega\}\chi)$ are mapped onto VCS basis states. For $[\xi_1\xi_2]=[\Xi_1\Xi_2]$, $\langle\omega\rangle=\langle\Omega+2\rangle$, and $\Xi_1 \neq |\Xi_2|$, it is necessary to determine the matrices $\mathcal{K}([\Xi_1\Xi_2]\{\Omega+2\})$ and $\mathcal{K}^{-1}([\Xi_1\Xi_2]\{\Omega+2\})$ by diagonalizing the 2×2 matrix $\mathcal{K}\mathcal{K}^{\dagger}([\Xi_1\Xi_2]\{\Omega+2\})$ and using (I5.16) and (I5.17). The eigenvalues of the matrix are given by

$$d_{1,2}=(2\Omega)^{-1}\{(\Omega+1)[2\Omega(\Omega+2)-C_1(\Xi_1,\Xi_2)]\pm\Delta\} \tag{4.17}$$

where d_1 (respectively, d_2) corresponds to the + (respectively, −) sign, and

$$\Delta=\{(\Omega+1)^2[C_1{}^2(\Xi_1,\Xi_2)-4C_2{}^2(\Xi_1,\Xi_2)]+4C_2{}^2(\Xi_1,\Xi_2)\}^{1/2}. \tag{4.18}$$

The (real) unitary matrix $\mathbf{U}$ converting $\mathcal{K}\mathcal{K}^{\dagger}([\Xi_1\Xi_2]\{\Omega+2\})$ to diagonal form can be written as

$$\mathbf{U}=\begin{pmatrix} -\cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{4.19}$$

where

$$\begin{aligned} \cos\phi &= (\Omega+1)\{2\Delta[2C_2(\Xi_1,\Xi_2)+\Delta]\}^{-1/2}[C_1{}^2(\Xi_1,\Xi_2)-4C_2{}^2(\Xi_1,\Xi_2)]^{1/2} \\ \sin\phi &= (2\Delta)^{-1/2}[2C_2(\Xi_1,\Xi_2)+\Delta]^{1/2}. \end{aligned} \tag{4.20}$$

Note that, when $\Xi_1=\pm\Xi_2$, the matrix $\mathcal{K}\mathcal{K}^{\dagger}([\Xi_1\Xi_2]\{\Omega+2\})$ is one dimensional and given by

$$\mathcal{K}\mathcal{K}^{\dagger}([\Xi_1\pm\Xi_1]\{\Omega+2\})=\Omega^{-1}(\Omega+2)(\Omega-\Xi_1)(\Omega+\Xi_1+1) \tag{4.21}$$

hence it coincides with d_2, as given by (4.17).

If $\Omega=\Xi_1$, then, as shown in column 6 of table 2, eight linear combinations of the sixteen VBB basis states $|[\lambda_1\lambda_2][\xi_1\xi_2]\langle\omega\rangle\{\omega\}\chi)$ are mapped onto the null vector whenever $\Xi_1 \neq \pm\Xi_2, \pm\Xi_2+1$. For $[\xi_1\xi_2]=[\Xi_1\Xi_2]$ and $\langle\omega\rangle=\langle\Omega+2\rangle$, the eigenvalue d_2 vanishes, so that only the eigenvector of $\mathcal{K}\mathcal{K}^{\dagger}([\Xi_1\Xi_2]\{\Omega+2\})$ corresponding to

$$d_1=\Xi_1{}^{-1}(\Xi_1+1)[\Xi_1(\Xi_1+1)-\Xi_2{}^2] \tag{4.22}$$

has to be retained whenever $\Xi_1 \neq |\Xi_2|$. According to (I5.18), (4.15), (4.19) and (4.20), it is given by

$$\begin{aligned} &|1[\Xi_1\Xi_2]\langle\Xi_1+2\rangle\{\Xi_1+2\}\chi) \\ &\quad=[2C_1(\Xi_1,\Xi_2)]^{-1/2}\{-[(\Xi_1-\Xi_2)(\Xi_1+\Xi_2+2)]^{1/2} \\ &\quad\quad\times|[11][\Xi_1\Xi_2]\langle\Xi_1+2\rangle\{\Xi_1+2\}\chi) \\ &\quad\quad+[(\Xi_1+\Xi_2)(\Xi_1-\Xi_2+2)]^{1/2}|[1-1][\Xi_1\Xi_2]\langle\Xi_1+2\rangle\{\Xi_1+2\}\chi)\}. \end{aligned} \tag{4.23}$$

In general, the branching rule for the decomposition of $[\Xi_1\Xi_2\Omega\rangle$ into $\mathrm{so}(4)\oplus\mathrm{sp}(2,\mathbb{R})$ irreps is obtained by combining the VBB and VCS conditions listed in columns 5 and 6 of table 2. In particular, when $\Omega=\Xi_1=\pm\Xi_2$, it results from the remark following (4.21) that no state with $[\xi_1\xi_2]=[\Xi_1\Xi_2]$ and $\langle\omega\rangle=\langle\Omega+2\rangle$ is allowed.

In a recent work (Schmitt *et al* 1989), the branching rule for the decomposition of $[\Xi_1\Xi_2\Omega\rangle$ into so(4)⊕sp(2, ℝ) irreps was determined by constructing and orthonormalizing the lowest-weight states of such irreps in a super Fock space. The results displayed in table 2 of the present paper are in complete agreement with the poles of the lowest-weight state normalization factors, given in appendix B of Schmitt *et al* (1989) (however, not with the branching rule given in (3.2) of the same reference, where there seems to be a misprint).

By applying (I5.22), (I5.23), (I5.35) and (I5.36), the so(4)⊕sp(2, ℝ) (triple) reduced matrix elements of the odd generators $\mathfrak{T} = (\mathfrak{H}, \mathfrak{J})$ can be finally obtained. They are listed in appendix 4 both for the generic case where $\Omega > \Xi_1 > |\Xi_2| + 1$, and for the special cases where not all of these conditions are fulfilled.

5. The osp(2/2N, ℝ) superalgebras

5.1. General remarks

The even part of the osp(2/2N, ℝ) superalgebra is the so(2)⊕sp(2N, ℝ) algebra, where sp(2N, ℝ) is generated by $D^{\dagger}_{ij}$, D^{ij}, $E_i{}^j$, $i, j = 1, \ldots, N$, and so(2) is spanned by C. As reviewed in section 7 of I, the odd raising (lowering) generators I_i and H_i (G^i and J^i) are the components of two separate so(2)⊕u(N) irreducible tensors $\boldsymbol{I}$ and $\boldsymbol{H}$ ($\boldsymbol{G}$ and $\boldsymbol{J}$), and the same is true for the Grassmann variables σ_i and τ_i (and their corresponding differential operators $\partial/\partial\sigma_i$ and $\partial/\partial\tau_i$). The positive discrete series irreps of osp(2/2N, ℝ) are characterized by $[\Xi\Omega\rangle = [\Xi\Omega_1\Omega_2\ldots\Omega_N\rangle$, where $\Xi \in \mathbb{Z}$, and $\Omega_1, \ldots, \Omega_N$ are integers satisfying the inequalities $\Omega_1 \geqslant \Omega_2 \geqslant \ldots \geqslant \Omega_N > N$. We shall consider here both star and grade star irreps corresponding to the adjoint relations $I_i = \pm(G^i)^{\dagger}$, $H_i = \pm(J^i)^{\dagger}$, or to the grade adjoint relations $I_i = \mp(G^i)^{\ddagger}$, $H_i = \pm(J^i)^{\ddagger}$.

The Q polynomials are characterized by an so(2) irrep $[\lambda]$, a u(N) irrep $\{\mu\} = \{\mu_1\mu_2\ldots\mu_N\}$, and a row index γ. Here $2 \geqslant \mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_N \geqslant 0$, $[\lambda]$ may run over those so(2) irreps contained in the u(2) irrep $\{\tilde{\mu}\} = \{\tilde{\mu}_1\tilde{\mu}_2\}$, i.e. $\lambda = \tilde{\mu}_1 - \tilde{\mu}_2, \tilde{\mu}_1 - \tilde{\mu}_2 - 2, \ldots, -\tilde{\mu}_1 + \tilde{\mu}_2$, and γ may be taken as (μ), where (μ) denotes a Gel'fand pattern of $\{\mu\}$. Hence the allowed $\{\mu\}$ irreps may be denoted by $\{\mu\} = \{2^k 1^{l-2k}\dot{0}\}$, where $0 \leqslant l \leqslant 2N$, $\max(0, l-N) \leqslant k \leqslant [\frac{1}{2}l]$, and $[\frac{1}{2}l]$ is the largest integer contained in $\frac{1}{2}l$. For such an irrep, λ runs over the range $l-2k, l-2k-2, \ldots, 2k-l$.

For given $\{\mu\}$ and $[\lambda]$ irreps, the degrees of Q in $\boldsymbol{\sigma}$ and in $\boldsymbol{\tau}$ are $\mathcal{N}_\sigma = \frac{1}{2}(l+\lambda)$, and $\mathcal{N}_\tau = \frac{1}{2}(l-\lambda)$, respectively. Hence, the Q polynomials may be constructed as follows:

$$Q^{[\lambda]\{\mu\}}_{(\mu)}(\boldsymbol{\sigma}, \boldsymbol{\tau}) = [Q^{[(\lambda-l)/2]\{1^{(l-\lambda)/2}\dot{0}\}}(\boldsymbol{\tau}) \times Q^{[(\lambda+l)/2]\{1^{(l+\lambda)/2}\dot{0}\}}(\boldsymbol{\sigma})]^{[\lambda]\cdot\mu\}}_{(\mu)} \tag{5.1}$$

by u(N) coupling a polynomial depending only on $\boldsymbol{\sigma}$ with another one depending only on $\boldsymbol{\tau}$. Both of the latter transform under an antisymmetric u(N) irrep, and their phase is fixed by choosing their highest-weight component as

$$Q^{[l]\{1^l\dot{0}\}}_{\text{hw}}(\boldsymbol{\sigma}) = \sigma_l \ldots \sigma_2\sigma_1 \qquad Q^{[-l]\{1^l\dot{0}\}}_{\text{hw}}(\boldsymbol{\tau}) = \tau_l \ldots \tau_2\tau_1. \tag{5.2}$$

From (5.1), we find that the factors appearing in (I7.14) and (I7.15) are $v_2 = -\sqrt{2}$ and $w_2 = -1$, respectively.

Standard u(N) tensor calculus (Le Blanc and Hecht 1987) enables one to obtain the u(N) reduced matrix elements of $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ between two polynomials (5.1) from those between two polynomials depending only on $\boldsymbol{\sigma}$ or $\boldsymbol{\tau}$. The latter are given by

$$([l+1]\{1^{l+1}\dot{0}\}\|\boldsymbol{\sigma}\|[l]\{1^{l}\dot{0}\}) = ([-l-1]\{1^{l+1}\dot{0}\}\|\boldsymbol{\tau}\|[-l]\{1^{l}\dot{0}\}) = \sqrt{l+1} \tag{5.3}$$

and the former by

$$\begin{aligned}
&([\lambda+1]\{2^{k+1}1^{l-2k-1}\dot{0}\}\|\boldsymbol{\sigma}\|[\lambda]\{2^{k}1^{l-2k}\dot{0}\})\\
&\qquad = (-1)^{(l-\lambda)/2}[2(l-2k)]^{-1/2}[(k+1)(l-\lambda-2k)]^{1/2}\\
&([\lambda+1]\{2^{k}1^{l-2k+1}\dot{0}\}\|\boldsymbol{\sigma}\|[\lambda]\{2^{k}1^{l-2k}\dot{0}\})\\
&\qquad = (-1)^{k}[2(l-2k+2)]^{-1/2}[(l-k+2)(l+\lambda-2k+2)]^{1/2}\\
&([\lambda-1]\{2^{k+1}1^{l-2k-1}\dot{0}\}\|\boldsymbol{\tau}\|[\lambda]\{2^{k}1^{l-2k}\dot{0}\})\\
&\qquad = (-1)^{(l-\lambda-2k)/2}[2(l-2k)]^{-1/2}[(k+1)(l+\lambda-2k)]^{1/2}\\
&([\lambda-1]\{2^{k}1^{l-2k+1}\dot{0}\}\|\boldsymbol{\tau}\|[\lambda]\{2^{k}1^{l-2k}\dot{0}\})\\
&\qquad = [2(l-2k+2)]^{-1/2}[(l-k+2)(l-\lambda-2k+2)]^{1/2}.
\end{aligned} \tag{5.4}$$

From (5.1) and (5.4), we also obtain the following reduced matrix elements:

$$\begin{aligned}
&([\lambda]\{2^{k+1}1^{l-2k}\dot{0}\}\|Q^{[0]\{2\dot{0}\}}(\boldsymbol{\sigma},\boldsymbol{\tau})\|[\lambda]\{2^{k}1^{l-2k}\dot{0}\})\\
&\qquad = (-1)^{(l-\lambda)/2}[\tfrac{1}{2}(k+1)(l-k+2)]^{1/2}
\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}
&([\lambda]\{2^{k+2}1^{l-2k-2}\dot{0}\}\|Q^{[0]\{1^2\dot{0}\}}(\boldsymbol{\sigma},\boldsymbol{\tau})\|[\lambda]\{2^{k}1^{l-2k}\dot{0}\})\\
&\qquad = (-1)^{k+1}\tfrac{1}{2}[(l-2k-1)(l-2k)]^{-1/2}\\
&\qquad\quad \times[(k+1)(k+2)(l+\lambda-2k)(l-\lambda-2k)]^{1/2}\\
&([\lambda]\{2^{k+1}1^{l-2k}\dot{0}\}\|Q^{[0]\{1^2\dot{0}\}}(\boldsymbol{\sigma},\boldsymbol{\tau})\|[\lambda]\{2^{k}1^{l-2k}\dot{0}\})\\
&\qquad = (-1)^{(l-\lambda+2)/2}\lambda[2(l-2k)(l-2k+2)]^{-1/2}[(k+1)(l-k+2)]^{1/2}\\
&([\lambda]\{2^{k}1^{l-2k+2}\dot{0}\}\|Q^{[0]\{1^2\dot{0}\}}(\boldsymbol{\sigma},\boldsymbol{\tau})\|[\lambda]\{2^{k}1^{l-2k}\dot{0}\})\\
&\qquad = (-1)^{k}\tfrac{1}{2}[(l-2k+2)(l-2k+3)]^{-1/2}\\
&\qquad\quad \times[(l-k+2)(l-k+3)(l+\lambda-2k+2)(l-\lambda-2k+2)]^{1/2}.
\end{aligned} \tag{5.6}$$

From the latter and (I6.22), (I6.26) and (I7.14), we can then determine the reduced matrix elements of $\Gamma^{(0)}(\boldsymbol{D}^{+})$ appearing in the recursion relations (I7.5) and (I7.6).

Hence, the only unknown quantities equations (I7.5) and (I7.6) may still contain are u(N) Racah coefficients of the type $U(\{\Omega\}\{\mu\}\{\omega'\}\{\mu''\};\,\{\omega\}\zeta\{\mu'\}\zeta')$, where $\{\mu''\} = \{1\dot{0}\}$, $\{2\dot{0}\}$ or $\{1^2\dot{0}\}$, since the latter appear in relations such as (I5.24) and (I6.26). In the next two subsections, we shall consider two examples wherein they are known, and hence the recursion relations for $\mathcal{H}\mathcal{H}^{+}$ can be explicitly written down and solved.

5.2. The most degenerate irreps of osp(2/2N, ℝ)

Whenever $\Omega_1 = \ldots = \Omega_N = \Omega$ so that the osp(2/2N, ℝ) irrep may be denoted by $[\Xi\dot{\Omega}\rangle$, the above-mentioned Racah coefficients are equal to 1 for allowed u(N) irreps, and 0 otherwise. In such a case, the orthonormal VBB basis states reducing the stability subalgebra so(2)⊕u(N) can be written in shorthand notation as $|[\xi]\langle\omega\rangle\{\nu\}\rho\{h\}\chi\rangle$, where the allowed so(2) and sp(2N, ℝ) irreps are, respectively

$$[\xi] = [\Xi + \lambda] \qquad \langle\omega\rangle = \langle(\Omega+2)^k(\Omega+1)^{l-2k}\dot{\Omega}\rangle \qquad 0 \leq l \leq 2N$$
$$\max(0, l-N) \leq k \leq [\tfrac{1}{2}l] \qquad \lambda = l-2k, l-2k-2, \ldots, 2k-l. \tag{5.7}$$

The irreps $[\lambda]$ and $\{\mu\} = \{2^k 1^{l-2k}\dot{0}\}$, entirely determined by $[\xi]$ and $\{\omega\}$, have been dropped as well as the unneeded multiplicity label ζ.

Since their rows and columns are labelled by $t = [\lambda]\{\mu\}$, all $\mathcal{K}([\xi]\{\omega\})$ submatrices are one dimensional. The recursion relations (I7.5) and (I7.6), satisfied by $\mathcal{K}\mathcal{K}^{\dagger}([\xi]\{\omega\})$, take the following simple form:

$$\mathcal{K}\mathcal{K}^{\dagger}([\xi \pm 1]\{\omega + \Delta^{(1)}(i)\})$$
$$= \delta_{\pm}[2(2\Omega - l + 1)]^{-1}(\Omega + \omega_i - i + 2)(2\Omega \mp \Xi \mp \xi - l)\mathcal{K}\mathcal{K}^{\dagger}([\xi]\{\omega\}) \tag{5.8}$$
$$i = k+1, l-k+1$$

where

$$\delta_{+} = \delta_{-} = \pm 1 \tag{5.9}$$

or

$$\delta_{+} = -\delta_{-} = \pm(-1)^{l} \tag{5.10}$$

according to whether one considers star or grade star irreps.

Let us first review the case of star irreps. The lower sign choice in (5.9) can be immediately ruled out, because the condition

$$\mathcal{K}\mathcal{K}^{\dagger}([\Xi \pm 1]\{\Omega + 1\dot{\Omega}\}) = -(\Omega \mp \Xi)\mathcal{K}\mathcal{K}^{\dagger}([\Xi]\{\dot{\Omega}\}) \geq 0 \tag{5.11}$$

cannot be fulfilled. For the upper sign choice, we obtain the positive semi-definite solution

$$\mathcal{K}\mathcal{K}^{\dagger}([\xi]\{\omega\})$$
$$= \left(\prod_{s=1}^{k} (2\Omega + s - l + 1)^{-1}(2\Omega - s + 3)\right)$$
$$\times \left(\prod_{s=1}^{(l-\lambda)/2} (\Omega + \Xi - s + 1)\right)\left(\prod_{s=1}^{(l+\lambda)/2} (\Omega - \Xi - s + 1)\right) \tag{5.12}$$

where $[\xi]$ and $\{\omega\}$ are given in (5.7), if and only if the condition

$$\Omega \geq |\Xi| \tag{5.13}$$

is satisfied. The irrep $[\Xi\dot{\Omega}\rangle$ is then equivalent to a star representation.

From (5.12), is clear that if $\Omega - N + 1 > |\Xi|$, then all the VBB basis states $|[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi\rangle$, corresponding to the allowed irreps (5.7), are mapped onto VCS basis states. In such a case, the branching rule for the decomposition of $[\Xi\dot{\Omega}\rangle$ into a direct sum of $\mathrm{so}(2)\oplus\mathrm{sp}(2N,\mathbb{R})$ irreps $[\xi]\oplus\langle\omega\rangle$ is given by

$$[\Xi\dot{\Omega}\rangle \downarrow \sum_{l=0}^{2N} \sum_{k=\max(0,l-N)}^{[l/2]} \sum_{\lambda=2k-l}^{l-2k}{}' \oplus([\Xi+\lambda]\oplus\langle(\Omega+2)^k(\Omega+1)^{l-2k}\dot{\Omega}\rangle) \tag{5.14}$$

where the prime on the summation symbol over λ means that the summation only runs over even or odd numbers according to the parity of $l-2k$.

On the contrary, if $\Omega - q = |\Xi|$, where q is some integer such that $0 \leqslant q \leqslant N-1$, then the VBB basis states $|[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi\rangle$ corresponding to l, k and λ values satisfying the conditions

$$l \geqslant q+1 \qquad k \leqslant l-q-1 \qquad \pm\lambda \geqslant 2q+2-l \tag{5.15}$$

are mapped onto the null vector. In (5.15), the upper (lower) sign applies to the case where $\Omega - q = \Xi$ $(-\Xi)$. The corresponding $\mathrm{so}(2)\oplus\mathrm{sp}(2N,\mathbb{R})$ irreps have to be eliminated from the branching rule (5.14), which therefore becomes

$$\begin{aligned}[\pm(\Omega-q)\dot{\Omega}\rangle \downarrow & \sum_{l=0}^{q} \sum_{k=0}^{[l/2]} \sum_{\lambda=2k-l}^{l-2k}{}' \oplus([\pm(\Omega-q+\lambda)]\oplus\langle(\Omega+2)^k(\Omega+1)^{l-2k}\dot{\Omega}\rangle) \\ & + \sum_{l=q+1}^{2q} \sum_{k=l-q}^{[l/2]} \sum_{\lambda=2k-l}^{l-2k}{}' \oplus([\pm(\Omega-q+\lambda)]\oplus\langle(\Omega+2)^k(\Omega+1)^{l-2k}\dot{\Omega}\rangle) \\ & + \sum_{l=q+1}^{N+q} \sum_{k=\max(0,l-N)}^{\min(q,l-q-1)} \sum_{\lambda=2k-l}^{2q-l}{}' \oplus([\pm(\Omega-q+\lambda)]\oplus\langle(\Omega+2)^k(\Omega+1)^{l-2k}\dot{\Omega}\rangle).\end{aligned} \tag{5.16}$$

Finally, by using a relation similar to (2.14), the $\mathrm{u}(N)$ reduced matrix elements (I7.9) and (I7.10) of the odd generators between two lowest-weight $\mathrm{so}(2)\oplus\mathrm{u}(N)$ irrep basis states can be easily obtained. In appendix 5 they are listed for the generic case where $\Omega - N + 1 > |\Xi|$. The results also apply whenever $\Omega - q = |\Xi|$, $0 \leqslant q \leqslant N-1$, provided only the allowed VCS basis states are retained.

In the $\mathrm{osp}(2/2,\mathbb{R})$ case, the irreps $[\Xi\Omega\rangle$ considered in the present subsection are the most general positive discrete series irreps. The solution (5.12) becomes

$$\begin{aligned}\mathcal{K}\mathcal{K}^{+}([\Xi\pm1]\{\Omega+1\}) &= \Omega\mp\Xi \\ \mathcal{K}\mathcal{K}^{+}([\Xi]\{\Omega+2\}) &= \Omega^{-1}(\Omega+1)(\Omega-\Xi)(\Omega+\Xi)\end{aligned} \tag{5.17}$$

for $\Omega \geqslant |\Xi|$. The branching rule for the decomposition of $[\Xi\Omega\rangle$ into $\mathrm{so}(2)\oplus\mathrm{sp}(2,\mathbb{R})$ irreps can be written as

$$[\Xi\Omega\rangle \downarrow \begin{cases} ([\Xi]\oplus\langle\Omega\rangle)\oplus([\Xi+1]\oplus\langle\Omega+1\rangle)\oplus([\Xi-1]\oplus\langle\Omega+1\rangle) & \\ \qquad \oplus([\Xi]\oplus\langle\Omega+2\rangle) & \text{if } \Omega > |\Xi| \\ ([\Xi]\oplus\langle\Omega\rangle)\oplus([\Xi\mp1]\oplus\langle\Omega+1\rangle) & \text{if } \Omega = \pm\Xi. \end{cases} \tag{5.18}$$

From the $\mathrm{u}(1)$ reduced matrix elements of the odd generators given in appendix 5, $\mathrm{sp}(2,\mathbb{R})$ (triple) reduced matrix elements can be obtained by applying (I5.35) and (I5.36) to the two separate $\mathrm{sp}(2,\mathbb{R})$ irreducible tensors $\mathcal{J} = (I, J)$ and $\mathcal{H} = (H, G)$. The results are also listed in appendix 5. They agree with those obtained by Balantekin *et al* (1989) from a direct resolution of the supercommutation relations (except for a sign in the fifth relation contained in their equation (A.8*a*), where there is a misprint).

Let us next review the case of grade star irreps. According to (5.7), for any N value there are VBB basis states $|[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi)$ characterized by the so(2)⊕u(N) irreps $[\xi]\oplus\{\omega\}=[\Xi]\oplus\{\dot{\Omega}\}$, $[\Xi\pm 1]\oplus\{\Omega+1\dot{\Omega}\}$, and $[\Xi]\oplus\{\Omega+2\dot{\Omega}\}$. From (5.8), they give rise to the following recursion relations:

$$
\begin{aligned}
&\mathcal{K}\mathcal{K}^{+}([\Xi+1]\{\Omega+1\dot{\Omega}\})=\pm(\Omega-\Xi)\mathcal{K}\mathcal{K}^{+}([\Xi]\{\dot{\Omega}\})\\
&\mathcal{K}\mathcal{K}^{+}([\Xi-1]\{\Omega+1\dot{\Omega}\})=\mp(\Omega+\Xi)\mathcal{K}\mathcal{K}^{+}([\Xi]\{\dot{\Omega}\})\\
&\mathcal{K}\mathcal{K}^{+}([\Xi]\{\Omega+2\dot{\Omega}\})=\pm\Omega^{-1}(\Omega+1)(\Omega+\Xi)\mathcal{K}\mathcal{K}^{+}([\Xi+1]\{\Omega+1\dot{\Omega}\})\\
&\mathcal{K}\mathcal{K}^{+}([\Xi]\{\Omega+2\dot{\Omega}\})=\mp\Omega^{-1}(\Omega+1)(\Omega-\Xi)\mathcal{K}\mathcal{K}^{+}([\Xi-1]\{\Omega+1\dot{\Omega}\})
\end{aligned}
\tag{5.19}
$$

where the upper (lower) signs correspond to the upper (lower) sign in (5.10). Equations (5.19) have the positive semi-definite solution

$$
\begin{aligned}
&\mathcal{K}\mathcal{K}^{+}([\Xi\pm 1]\{\Omega+1\dot{\Omega}\})=\Omega\mp\Xi=2\Omega\\
&\mathcal{K}\mathcal{K}^{+}([\Xi\mp 1]\{\Omega+1\dot{\Omega}\})=\mathcal{K}\mathcal{K}^{+}([\Xi]\{\Omega+2\dot{\Omega}\})=0
\end{aligned}
\tag{5.20}
$$

if and only if $\Omega=\mp\Xi$. For $N=1$, there are no other allowed VBB basis states $|[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\chi)$. For $N\geqslant 2$, there are some, among which are states corresponding to the irreps $[\Xi\pm 2]\oplus\{(\Omega+1)^2\dot{\Omega}\}$. The latter give rise to the relation

$$
\mathcal{K}\mathcal{K}^{+}([\Xi\pm 2]\{(\Omega+1)^2\dot{\Omega}\})=-(\Omega\mp\Xi-1)\mathcal{K}\mathcal{K}^{+}([\Xi\pm 1]\{\Omega+1\dot{\Omega}\}).\tag{5.21}
$$

For $\Omega=\mp\Xi$, the right-hand side of (5.21) is negative definite. We conclude that the only osp(2/2N, ℝ) irreps $[\Xi\dot{\Omega}\rangle$, which are equivalent to grade star representations, are the irreps $[-\Omega\Omega\rangle$ or $[\Omega\Omega\rangle$ of osp(2/2, ℝ), according as one chooses the upper or lower sign in (5.10).

For such irreps, it results from (I7.9), (I7.10) and from relations similar to (I5.35) and (I5.36) that the sp(2, ℝ) reduced matrix elements of $\mathcal{J}$ are given by

$$
([-\Omega+1]\langle\Omega+1\rangle|||\gamma(\mathcal{J})|||[-\Omega]\langle\Omega\rangle)=-[2(\Omega-1)]^{1/2}\tag{5.22}
$$

and

$$
([\Omega]\langle\Omega\rangle|||\gamma(\mathcal{J})|||[\Omega-1]\langle\Omega+1\rangle)=(2\Omega)^{1/2}.\tag{5.23}
$$

Comparison with (A5.4) and (A5.6), where $\Xi=-\Omega$ and $\Xi=\Omega$ respectively, shows that the osp(2/2, ℝ) irreps $[-\Omega\Omega\rangle$ and $[\Omega\Omega\rangle$ are at the same time equivalent to both star and grade star representations.

5.3. The osp(2/4, ℝ) superalgebra

For the osp(2/4, ℝ) irreps $[\Xi\Omega_1\Omega_2\rangle$, all reduced matrix elements can be expressed in terms of u(2) Racah coefficients, so that the recursion relation for $\mathcal{K}\mathcal{K}^{+}$ can again be explicitly written down. Since the case where $\Omega_1=\Omega_2$ was treated in the previous subsection, we shall assume here that Ω_1 is greater than Ω_2.

In shorthand notation, the orthonormal VBB basis states reducing the stability subalgebra so(2)⊕u(2) can be written as $|\{\mu_1\mu_2\}[\xi]\langle\omega_1\omega_2\rangle\{\nu_1\nu_2\}\{h_1h_2\}\chi)$, since the so(2) irrep $[\lambda]$ is determined by $[\xi]$ through the relation

$$
\lambda=\xi-\Xi\tag{5.24}
$$

and all couplings are multiplicity free. The allowed irreps $[\xi]$ and $\langle\omega_1\omega_2\rangle$ are listed in columns 1 and 2 of table 3, and the conditions for their existence are displayed in column 5 of the same table. The latter result from the coupling rule of the angular momenta $\frac{1}{2}(\Omega_1-\Omega_2)$ and $\frac{1}{2}(\mu_1-\mu_2)$ to total angular momentum $\frac{1}{2}(\omega_1-\omega_2)$.

Table 3. Branching rule for the decomposition of an osp(2/4, ℝ) star irrep $[\Xi\Omega_1\Omega_2\rangle$ with $\Omega_1 > \Omega_2$ into so(2)⊕sp(4, ℝ) irreps $[\xi]\oplus\langle\omega_1\omega_2\rangle$.

$[\xi]$	$\langle\omega_1\omega_2\rangle$	$[\lambda]$	$\{\mu_1\mu_2\}$	VBB conditions	VCS conditions
$[\Xi]$	$\langle\Omega_1\Omega_2\rangle$	[0]	{00}	—	—
$[\Xi+1]$	$\langle\Omega_1+1\Omega_2\rangle$	[1]	{10}	—	—
$[\Xi+1]$	$\langle\Omega_1\Omega_2+1\rangle$	[1]	{10}	—	$\Omega_2-1\neq\Xi$
$[\Xi-1]$	$\langle\Omega_1+1\Omega_2\rangle$	[−1]	{10}	—	—
$[\Xi-1]$	$\langle\Omega_1\Omega_2+1\rangle$	[−1]	{10}	—	$\Omega_2-1\neq-\Xi$
$[\Xi+2]$	$\langle\Omega_1+1\Omega_2+1\rangle$	[2]	{11}	—	$\Omega_2-1\neq\Xi$
$[\Xi]$	$\langle\Omega_1+2\Omega_2\rangle$	[0]	{20}	—	—
$[\Xi]$	$\langle\Omega_1+1\Omega_2+1\rangle$	[0]	{20}	—	—†
		[0]	{11}	—	$\Omega_2-1\neq\Xi,-\Xi$‡
$[\Xi]$	$\langle\Omega_1\Omega_2+2\rangle$	[0]	{20}	$\Omega_1\neq\Omega_2+1$	$\Omega_2-1\neq\Xi,-\Xi$
$[\Xi-2]$	$\langle\Omega_1+1\Omega_2+1\rangle$	[−2]	{11}	—	$\Omega_2-1\neq-\Xi$
$[\Xi+1]$	$\langle\Omega_1+2\Omega_2+1\rangle$	[1]	{21}	—	$\Omega_2-1\neq\Xi$
$[\Xi+1]$	$\langle\Omega_1+1\Omega_2+2\rangle$	[1]	{21}	—	$\Omega_2-1\neq\Xi,-\Xi$
$[\Xi-1]$	$\langle\Omega_1+2\Omega_2+1\rangle$	[−1]	{21}	—	$\Omega_2-1\neq-\Xi$
$[\Xi-1]$	$\langle\Omega_1+1\Omega_2+2\rangle$	[−1]	{21}	—	$\Omega_2-1\neq\Xi,-\Xi$
$[\Xi]$	$\langle\Omega_1+2\Omega_2+2\rangle$	[0]	{22}	—	$\Omega_2-1\neq\Xi,-\Xi$

† Condition valid for eigenvalue d_1.
‡ Condition valid for eigenvalue d_2.

Since their rows and columns are labelled by $t=[\lambda]\{\mu_1\mu_2\}$, all $\mathscr{K}([\xi]\{\omega_1\omega_2\})$ submatrices are one dimensional, except for $\mathscr{K}([\Xi]\{\Omega_1+1\Omega_2+1\})$ which is two dimensional. In the latter case, we shall abbreviate t by $t=\{20\}$ or $\{11\}$. The recursion relations (I7.5) and (I7.6) give rise to forty equations, which have to be satisfied by sixteen unknowns.

Let us first review the case of star irreps. Among the forty equations, let us quote the following four:

$$\begin{aligned}
\mathscr{K}\mathscr{K}^+([\Xi+1]\{\Omega_1+1\Omega_2\}) &= \pm(\Omega_1-\Xi)\mathscr{K}\mathscr{K}^+([\Xi]\{\Omega_1\Omega_2\})\\
\mathscr{K}\mathscr{K}^+([\Xi-1]\{\Omega_1+1\Omega_2\}) &= \pm(\Omega_1+\Xi)\mathscr{K}\mathscr{K}^+([\Xi]\{\Omega_1\Omega_2\})\\
\mathscr{K}\mathscr{K}^+([\Xi+1]\{\Omega_1\Omega_2+1\}) &= \pm(\Omega_2-\Xi-1)\mathscr{K}\mathscr{K}^+([\Xi]\{\Omega_1\Omega_2\})\\
\mathscr{K}\mathscr{K}^+([\Xi-1]\{\Omega_1\Omega_2+1\}) &= \pm(\Omega_2+\Xi-1)\mathscr{K}\mathscr{K}^+([\Xi]\{\Omega_1\Omega_2\})
\end{aligned} \tag{5.25}$$

corresponding to so(2)⊕u(2) irreps $[\xi]\oplus\{\omega_1\omega_2\}$ for which VBB basis states $|\{\mu_1\mu_2\}[\xi]\langle\omega_1\omega_2\rangle\{00\}\{\omega_1\omega_2\}\chi\rangle$ always exist. For the lower sign choice, these four equations have no positive semi-definite solution, whereas for the upper sign choice such a solution does exist if and only if

$$\Omega_2-1\geqslant|\Xi|. \tag{5.26}$$

It can be easily proved that in the latter case the remaining equations also have a positive semi-definite solution whenever condition (5.26) is satisfied.

The solution to the whole set of equations is

$$\mathscr{K}\mathscr{K}^+([\Xi\pm1]\{\Omega_1+1\Omega_2\})=\Omega_1\mp\Xi \qquad \mathscr{K}\mathscr{K}^+([\Xi\pm1]\{\Omega_1\Omega_2+1\})=\Omega_2\mp\Xi-1$$

$$\mathscr{K}\mathscr{K}^+([\Xi\pm2]\{\Omega_1+1\Omega_2+1\})=(\Omega_1\mp\Xi)(\Omega_2\mp\Xi-1)$$

$$\mathscr{K}\mathscr{K}^+([\Xi]\{\Omega_1+2\Omega_2\})=\Omega_1^{-1}(\Omega_1+1)(\Omega_1+\Xi)(\Omega_1-\Xi)$$

$$
\begin{aligned}
&(\mathscr{K}\mathscr{K}^{\dagger}([\Xi]\{\Omega_1+1\Omega_2+1\}))_{\{20\},\{20\}} \\
&\qquad =(\Omega_1+\Omega_2-2)^{-1}[\Omega_1(\Omega_2-1)(\Omega_1+\Omega_2-2)-(\Omega_1+\Omega_2)\Xi^2] \\
&(\mathscr{K}\mathscr{K}^{\dagger}([\Xi]\{\Omega_1+1\Omega_2+1\}))_{\{11\},\{11\}} \\
&\qquad =(\Omega_1+\Omega_2-2)^{-1}[\Omega_1(\Omega_2-1)(\Omega_1+\Omega_2)-(\Omega_1+\Omega_2-2)\Xi^2] \\
&(\mathscr{K}\mathscr{K}^{\dagger}([\Xi]\{\Omega_1+1\Omega_2+1\}))_{\{20\},\{11\}} \\
&\qquad =(\Omega_1+\Omega_2-2)^{-1}(\Omega_1+\Omega_2-1)\Xi[(\Omega_1-\Omega_2)(\Omega_1-\Omega_2+2)]^{1/2} \\
&\mathscr{K}\mathscr{K}^{\dagger}([\Xi]\{\Omega_1\Omega_2+2\})=(\Omega_2-1)^{-1}\Omega_2(\Omega_2+\Xi-1)(\Omega_2-\Xi-1) \\
&\mathscr{K}\mathscr{K}^{\dagger}([\Xi\pm 1]\{\Omega_1+2\Omega_2+1\}) \\
&\qquad =[\Omega_1(\Omega_1+\Omega_2-1)]^{-1}(\Omega_1+1)(\Omega_1+\Omega_2)(\Omega_1+\Xi)(\Omega_1-\Xi)(\Omega_2\mp\Xi-1) \\
&\mathscr{K}\mathscr{K}^{\dagger}([\Xi\pm 1]\{\Omega_1+1\Omega_2+2\}) \\
&\qquad =[(\Omega_2-1)(\Omega_1+\Omega_2-1)]^{-1}\Omega_2(\Omega_1+\Omega_2)(\Omega_1\mp\Xi)(\Omega_2+\Xi-1)(\Omega_2-\Xi-1) \\
&\mathscr{K}\mathscr{K}^{\dagger}([\Xi]\{\Omega_1+2\Omega_2+2\}) \\
&\qquad =[\Omega_1(\Omega_2-1)(\Omega_1+\Omega_2-1)]^{-1}(\Omega_1+1)\Omega_2(\Omega_1+\Omega_2+1) \\
&\qquad\quad \times(\Omega_1+\Xi)(\Omega_1-\Xi)(\Omega_2+\Xi-1)(\Omega_2-\Xi-1).
\end{aligned}
\tag{5.27}
$$

To check that the 2×2 matrix $\mathscr{K}\mathscr{K}^{\dagger}([\Xi]\{\Omega_1+1\Omega_2+1\})$ is positive semi-definite whenever condition (5.26) is fulfilled, it is sufficient to show that its trace and its determinant are not negative. From (5.26) and (5.27), it follows that

$$
\begin{aligned}
&\operatorname{tr}\mathscr{K}\mathscr{K}^{\dagger}([\Xi]\{\Omega_1+1\Omega_2+1\})=2(\Omega_1+\Omega_2-2)^{-1}(\Omega_1+\Omega_2-1)[\Omega_1(\Omega_2-1)-\Xi^2]\geqslant 0 \\
&\det\mathscr{K}\mathscr{K}^{\dagger}([\Xi]\{\Omega_1+1\Omega_2+1\}) \\
&\qquad =(\Omega_1+\Omega_2-2)^{-1}(\Omega_1+\Omega_2)(\Omega_1{}^2-\Xi^2)[(\Omega_2-1)^2-\Xi^2]\geqslant 0.
\end{aligned}
\tag{5.28}
$$

Whenever condition (5.26) is satisfied, and only in such a case, the irrep $[\Xi\Omega_1\Omega_2\rangle$ with $\Omega_1>\Omega_2$ is therefore equivalent to a star representation.

If $\Omega_2-1>|\Xi|$, then all the VBB basis states $|\{\mu_1\mu_2\}[\xi]\langle\omega_1\omega_2\rangle\{00\}\{\omega_1\omega_2\}\chi\rangle$ are mapped onto VCS basis states. For $[\xi]=[\Xi]$, $\langle\omega_1\omega_2\rangle=\langle\Omega_1+1\Omega_2+1\rangle$, it is necessary to determine the matrices $\mathscr{K}([\Xi]\{\Omega_1+1\Omega_2+1\})$ and $\mathscr{K}^{-1}([\Xi]\{\Omega_1+1\Omega_2+1\})$ by diagonalizing the 2×2 matrix $\mathscr{K}\mathscr{K}^{\dagger}([\Xi]\{\Omega_1+1\Omega_2+1\})$ and using (I5.16) and (I5.17). The eigenvalues of the latter can be expressed as

$$
d_{1,2}=(\Omega_1+\Omega_2-2)^{-1}\{(\Omega_1+\Omega_2-1)[\Omega_1(\Omega_2-1)-\Xi^2]\pm\Delta\}
\tag{5.29}
$$

where d_1 (respectively, d_2) corresponds to the + (respectively, −) sign, and

$$
\Delta=\{[\Omega_1(\Omega_2-1)+\Xi^2]^2+(\Omega_1-\Omega_2)(\Omega_1-\Omega_2+2)(\Omega_1+\Omega_2-1)^2\Xi^2\}^{1/2}.
\tag{5.30}
$$

The (real) unitary matrix **U** converting $\mathscr{K}\mathscr{K}^{\dagger}([\Xi]\{\Omega_1+1\Omega_2+1\})$ to diagonal form is given by (4.19), where

$$
\begin{aligned}
&\cos\phi=-(\Xi/|\Xi|)(2\Delta)^{-1/2}[-\Omega_1(\Omega_2-1)-\Xi^2+\Delta]^{1/2} \\
&\sin\phi=(2\Delta)^{-1/2}[\Omega_1(\Omega_2-1)+\Xi^2+\Delta]^{1/2}.
\end{aligned}
\tag{5.31}
$$

On the contrary, if $\Omega_2-1=|\Xi|$, then as shown in column 6 of table 3, some linear combinations of VBB basis states $|\{\mu_1\mu_2\}[\xi]\langle\omega_1\omega_2\rangle\{00\}\{\omega_1\omega_2\}\chi\rangle$ are mapped onto the null vector. For $[\xi]=[\Xi]$ and $\langle\omega_1\omega_2\rangle=\langle\Omega_1+1\Omega_2+1\rangle$, the eigenvalue d_2 vanishes, so that only the eigenvector of $\mathcal{K}\mathcal{K}^\dagger([\Xi]\{\Omega_1+1\Omega_2+1\})$ corresponding to

$$d_1=2(\Omega_1+\Omega_2-2)^{-1}(\Omega_2-1)(\Omega_1+\Omega_2-1)(\Omega_1-\Omega_2+1) \tag{5.32}$$

has to be retained. According to (I5.18), (4.19) and (5.31), it is given by

$$\begin{aligned}
&|1[\pm(\Omega_2-1)]\langle\Omega_1+1\Omega_2+1\rangle\{00\}\{\Omega_1+1\Omega_2+1\}\chi\rangle \\
&\quad=[2(\Omega_1-\Omega_2+1)]^{-1/2} \\
&\qquad\times\{\pm(\Omega_1-\Omega_2)^{1/2}|\{20\}[\pm(\Omega_2-1)]\langle\Omega_1+1\Omega_2+1\rangle\{00\}\{\Omega_1+1\Omega_2+1\}\chi\rangle \\
&\qquad+(\Omega_1-\Omega_2+2)^{1/2}|\{11\}[\pm(\Omega_2-1)]\langle\Omega_1+1\Omega_2+1\rangle\{00\}\{\Omega_1+1\Omega_2+1\}\chi\rangle\}.
\end{aligned} \tag{5.33}$$

In general, the branching rule for the decomposition of $[\Xi\Omega_1\Omega_2\rangle$ into so(2)⊕sp(4, ℝ) irreps can be obtained by combining the VBB and VCS conditions listed in columns 5 and 6 of table 3.

Finally, by applying (I7.10), the u(2) reduced matrix elements of the odd raising generators $\boldsymbol{I}$ and $\boldsymbol{H}$ between two lowest-weight so(2)⊕u(2) irrep basis states can be easily obtained. In appendix 6, they are listed both for the generic case where $\Omega_1>\Omega_2+1>|\Xi|+2$, and for the special cases where not all of these conditions are fulfilled.

Let us next review the case of grade star irreps. Among the forty equations satisfied by $\mathcal{K}\mathcal{K}^\dagger([\xi]\{\omega_1\omega_2\})$, let us quote the following four:

$$\begin{aligned}
&\mathcal{K}\mathcal{K}^\dagger([\Xi+1]\{\Omega_1+1\Omega_2\})=\pm(\Omega_1-\Xi)\mathcal{K}\mathcal{K}^\dagger([\Xi]\{\Omega_1\Omega_2\}) \\
&\mathcal{K}\mathcal{K}^\dagger([\Xi-1]\{\Omega_1+1\Omega_2\})=\mp(\Omega_1+\Xi)\mathcal{K}\mathcal{K}^\dagger([\Xi]\{\Omega_1\Omega_2\}) \\
&\mathcal{K}\mathcal{K}^\dagger([\Xi+2]\{\Omega_1+1\Omega_2+1\})=\mp(\Omega_2-\Xi-1)\mathcal{K}\mathcal{K}^\dagger([\Xi+1]\{\Omega_1+1\Omega_2\}) \\
&\mathcal{K}\mathcal{K}^\dagger([\Xi-2]\{\Omega_1+1\Omega_2+1\})=\pm(\Omega_2+\Xi-1)\mathcal{K}\mathcal{K}^\dagger([\Xi-1]\{\Omega_1+1\Omega_2\})
\end{aligned} \tag{5.34}$$

corresponding to so(2)⊕u(2) irreps $[\xi]\oplus\{\omega_1\omega_2\}$ for which VBB basis states $|\{\mu_1\mu_2\}[\xi]\langle\omega_1\omega_2\rangle\{00\}\{\omega_1\omega_2\}\chi\rangle$ always exist. No positive semi-definite solution does exist for any sign choice. We therefore conclude that no osp(2/4, ℝ) irrep $[\Xi\Omega_1\Omega_2\rangle$ with $\Omega_1>\Omega_2$ is equivalent to a grade star representation.

6. Concluding remarks

In the present paper, we did illustrate with various examples the power of the VCS and K-matrix combined theory for constructing matrix realizations of the osp($P/2N$, ℝ) positive discrete series irreps. The cases considered here are only part of those amenable to a full analytic treatment. Among them, let us mention the most general irreps of osp(3/4, ℝ) and osp(4/4, ℝ), for which the determination of reduced matrix elements only makes use of u(2) Racah coefficients. To provide all the information required to treat those cases as well, we presented the calculation techniques in a more general framework than was necessary for dealing with the examples actually worked out.

From the few examples here considered, some general trends already emerge very clearly. Considering grade star representations is almost useless. Such representations, which in principle might exist for any osp($2/2N$, ℝ) superalgebra, were in fact only found for osp(2/2, ℝ). In addition, the few grade star representations encountered are at the same time equivalent to star representations. On the contrary, a great variety of star representations were shown to exist. By analysing the results obtained for them, we may conjecture that a necessary condition for an osp($P/2N$, ℝ) positive discrete series irrep $[\Xi\Omega\rangle \equiv [\Xi_1 \ldots \Xi_M \Omega_1 \ldots \Omega_N\rangle$ to be a star representation is $\Omega_N \geqslant |\Xi_1|$, and that a necessary and sufficient condition for such an irrep to be typical is $\Omega_N - N + 1 > |\Xi_1|$.

In the present series of papers, we restricted ourselves to those osp($P/2N$, ℝ) irreps which can be induced from the direct sum of an so(p) (true) irrep and of an sp($2N$, ℝ) positive discrete series irrep. It is obvious that the same kind of techniques would apply to the cases of an so(p) spin irrep or of an sp($2N$, ℝ) negative discrete series irrep. Whether they may be used if one considers any harmonic series irrep of sp($2N$, ℝ) is not very clear yet. In particular, the case of the so-called mock-discrete irreps (King and Wybourne 1985), for which the VCS measure determined by Quesne (1986) is not valid (see also Perelomov 1977), would need some further investigation.

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Appendix 1. Reduced matrix elements of irreducible tensors for osp(1/2N, ℝ)

The purpose of the present appendix is to give explicit expressions for the reduced matrix elements of the irreducible tensors appearing in the recursion relation (I5.11) for $\mathcal{K}\mathcal{K}^{\dagger}(\{\omega\})$ in the osp($1/2N$, ℝ) case. Since $\Gamma_1^{(0)}(\boldsymbol{D}^{\dagger})$ vanishes and since the reduced matrix elements of $[\Gamma^{(0)}(\mathfrak{I})]^{\dagger}$, $\Gamma^{(0)}(\mathfrak{H})$, $\Gamma^{(1)}(\boldsymbol{D}^{\dagger})$ were given in (I6.14), (I6.15), (I6.20) respectively and assume an explicit form when (2.7) is taken into account, it only remains to consider the reduced matrix elements of $\Gamma^{(1)}(\mathfrak{H})$ and $\Gamma_2^{(0)}(\boldsymbol{D}^{\dagger})$.

By using (I5.25) and the relation $u_1 = 1$, as well as some results of Hecht *et al* (1981) and Le Blanc and Hecht (1987), equation (I6.21) can be written as

$$\begin{aligned}(\langle\omega''\rangle\{2\dot{0}\}\{\omega'\}\|\Gamma^{(1)}(\mathfrak{H})\|\langle\omega\rangle\{\dot{0}\}\{\omega\})\\ &= (-1)^{l-s}\Bigg[(\Omega_i - \Omega_{p_s} + p_s - i + 1)(\Omega_i - \Omega_{p_s} + p_s - i)^{-1}\\ &\quad\times\Bigg(\prod_{k\neq s}(\Omega_{p_k} - \Omega_{p_s} + p_s - p_k + 1)(\Omega_{p_k} - \Omega_{p_s} + p_s - p_k)^{-1}\Bigg)\Bigg]^{1/2}. \qquad \text{(A1.1)}\end{aligned}$$

Here $\{\omega\}$, $\{\omega'\}$ are given by (2.8), and $\{\omega''\}$ by (I5.13) where $j = p_s$, thence

$$\{\omega''\} = \{\Omega + \Delta^{(l-1)}(p_1 \ldots p_{s-1} p_{s+1} \ldots p_l)\}. \qquad \text{(A1.2)}$$

The reduced matrix element of $\Gamma_2^{(0)}(\boldsymbol{D}^{\dagger})$ is given by (I6.22), where, from (I6.12) and (2.1), it results that $w_1 = -1$. On the right-hand side of (I6.22), the only quantity remaining to be determined is the reduced matrix element $(\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|Q^{\{1^2\dot{0}\}}(\mathfrak{s})\|\langle\omega''\rangle\{\dot{0}\}\{\omega''\})$. In the present case, equation (I6.26) is useless for such a purpose because it contains an unknown u(N) Racah coefficient. This difficulty can be circumvented by writing $Q^{\{1^2\dot{0}\}}_{(i_1 i_2)}(\mathfrak{s})$ as (Biedenharn and Louck 1968)

$$Q^{\{1^2\dot{0}\}}_{(i_1 i_2)}(\mathfrak{s}) = \frac{1}{\sqrt{2}}[Q^{\{1\dot{0}\}}(\mathfrak{s}) \times Q^{\{1\dot{0}\}}(\mathfrak{s})]^{\{1^2\dot{0}\}}_{(i_1 i_2)}. \qquad \text{(A1.3)}$$

Then standard u(N) tensor calculus leads to the following result:

$$\begin{aligned}&(\langle\omega'\rangle\{\dot 0\}\{\omega'\}\|Q^{\{1^2\dot 0\}}(\mathfrak{s})\|\langle\omega''\rangle\{\dot 0\}\{\omega''\})\\&\quad=\frac{1}{\sqrt 2}\sum_{\{\bar\omega\}}U(\{\omega''\}\{1\dot 0\}\{\omega'\}\{1\dot 0\};\{\bar\omega\}\{1^2\dot 0\})(\langle\omega'\rangle\{\dot 0\}\{\omega'\}\|\mathfrak{s}\|\langle\bar\omega\rangle\{\dot 0\}\{\bar\omega\})\\&\quad\quad\times(\langle\bar\omega\rangle\{\dot 0\}\{\bar\omega\}\|\mathfrak{s}\|\langle\omega''\rangle\{\dot 0\}\{\omega''\})\end{aligned}\tag{A1.4}$$

where the summation runs over $\{\bar\omega\}=\{\omega\}$, $\{\omega+\Delta^{(1)}(i)-\Delta^{(1)}(j)\}$. By taking (2.7) and some results of Le Blanc and Hecht (1987) into account, equation (A1.4) can be rewritten as

$$\begin{aligned}&(\langle\omega'\rangle\{\dot 0\}\{\omega'\}\|Q^{\{1^2\dot 0\}}(\mathfrak{s})\|\langle\omega''\rangle\{\dot 0\}\{\omega''\})\\&\quad=S(p_s-i)(-1)^{m+s+1}\left(\prod_{k\neq s}(\Omega_{p_k}-\Omega_i+i-p_k+1)(\Omega_{p_k}-\Omega_{p_s}+p_s-p_k+1)\right.\\&\quad\quad\left.\times[(\Omega_{p_k}-\Omega_i+i-p_k)(\Omega_{p_k}-\Omega_{p_s}+p_s-p_k)]^{-1}\right)^{1/2}\end{aligned}\tag{A1.5}$$

where

$$S(j-i)=\begin{cases}1 & \text{if } j\geqslant i\\ -1 & \text{if } j<i.\end{cases}\tag{A1.6}$$

Introducing (A1.5) into (I6.22) and taking (I6.23) and (I6.25) into account, we finally obtain

$$\begin{aligned}&(\langle\omega'\rangle\{\dot 0\}\{\omega'\}\|\Gamma_2^{(0)}(\boldsymbol{D}^+)\|\langle\omega''\rangle\{\dot 0\}\{\omega''\})\\&\quad=S(p_s-i)(-1)^{m+s+1}\tfrac{1}{2}\Big[(\Omega_i-\Omega_{p_s}+p_s-i-1)(\Omega_i-\Omega_{p_s}+p_s-i+1)\\&\quad\quad\times\left(\prod_{k\neq s}(\Omega_{p_k}-\Omega_i+i-p_k+1)(\Omega_{p_k}-\Omega_{p_s}+p_s-p_k+1)\right.\\&\quad\quad\left.\times[(\Omega_{p_k}-\Omega_i+i-p_k)(\Omega_{p_k}-\Omega_{p_s}+p_s-p_k)]^{-1}\right)\Big]^{1/2}.\end{aligned}\tag{A1.7}$$

Appendix 2. Recursion relations for $\mathcal{HH}^+([\xi]\{\omega\})$ in the osp(3/2, $\mathbb{R}$) case

The thirteen recursion relations for $\mathcal{HH}^+([\xi]\{\omega\})$, where $[\xi]$ and $\{\omega\}$ run over the so(3) and u(1) irreps listed in columns 1 and 2 of table 1, assume the following form:

$$\mathcal{HH}^+([\Xi+1]\{\Omega+1\})=\pm(\Omega-\Xi)\mathcal{HH}^+([\Xi]\{\Omega\})\tag{A2.1}$$

$$\mathcal{HH}^+([\Xi]\{\Omega+1\})=\pm(\Omega+1)\mathcal{HH}^+([\Xi]\{\Omega\})\tag{A2.2}$$

$$\mathcal{HH}^+([\Xi-1]\{\Omega+1\})=\pm(\Omega+\Xi+1)\mathcal{HH}^+([\Xi]\{\Omega\})\tag{A2.3}$$

$$\mathcal{HH}^+([\Xi+1]\{\Omega+2\})=\pm(\Omega+1)\mathcal{HH}^+([\Xi+1]\{\Omega+1\})\tag{A2.4}$$

$$\mathcal{HH}^+([\Xi]\{\Omega+2\})=\pm\Omega^{-1}(\Omega+1)(\Omega+\Xi+1)\mathcal{HH}^+([\Xi+1]\{\Omega+1\})\tag{A2.5}$$

$$\mathcal{HH}^+([\Xi+1]\{\Omega+2\})=\pm(\Omega-\Xi)\mathcal{HH}^+([\Xi]\{\Omega+1\})\tag{A2.6}$$

$$\mathcal{HH}^+([\Xi]\{\Omega+2\})=\pm\Omega^{-1}(\Omega-\Xi)(\Omega+\Xi+1)\mathcal{HH}^+([\Xi]\{\Omega+1\})\tag{A2.7}$$

$$\mathcal{HK}^{+}([\Xi-1]\{\Omega+2\})=\pm(\Omega+\Xi+1)\mathcal{HK}^{+}([\Xi]\{\Omega+1\}) \quad (A2.8)$$

$$\mathcal{HK}^{+}([\Xi]\{\Omega+2\})=\pm\Omega^{-1}(\Omega+1)(\Omega-\Xi)\mathcal{HK}^{+}([\Xi-1]\{\Omega+1\}) \quad (A2.9)$$

$$\mathcal{HK}^{+}([\Xi-1]\{\Omega+2\})=\pm(\Omega+1)\mathcal{HK}^{+}([\Xi-1]\{\Omega+1\}) \quad (A2.10)$$

$$\mathcal{HK}^{+}([\Xi]\{\Omega+3\})=\pm(\Omega+1)^{-1}(\Omega+2)(\Omega+\Xi+1)\mathcal{HK}^{+}([\Xi+1]\{\Omega+2\}) \quad (A2.11)$$

$$\mathcal{HK}^{+}([\Xi]\{\Omega+3\})=\pm(\Omega+1)^{-1}\Omega(\Omega+2)\mathcal{HK}^{+}([\Xi]\{\Omega+2\}) \quad (A2.12)$$

$$\mathcal{HK}^{+}([\Xi]\{\Omega+3\})=\pm(\Omega+1)^{-1}(\Omega+2)(\Omega-\Xi)\mathcal{HK}^{+}([\Xi-1]\{\Omega+2\}). \quad (A2.13)$$

Appendix 3. Triple reduced matrix elements of the osp(3/2, ℝ) odd generators

In the generic case corresponding to $\Omega>\Xi>0$, the so(3)⊕sp(2, ℝ) (triple) reduced matrix elements $([\xi']\langle\omega+1\rangle|||\gamma(\mathfrak{X})|||[\xi]\langle\omega\rangle)$ of the osp(3/2, ℝ) odd generators are given by

$$([\Xi+1]\langle\Omega+1\rangle|||\gamma(\mathfrak{X})|||[\Xi]\langle\Omega\rangle)=-\Omega^{-1/2}[(\Omega-1)(\Omega-\Xi)]^{1/2} \quad (A3.1)$$

$$([\Xi]\langle\Omega+1\rangle|||\gamma(\mathfrak{X})|||[\Xi]\langle\Omega\rangle)=-\Omega^{-1/2}[(\Omega-1)(\Omega+1)]^{1/2} \quad (A3.2)$$

$$([\Xi-1]\langle\Omega+1\rangle|||\gamma(\mathfrak{X})|||[\Xi]\langle\Omega\rangle)=-\Omega^{-1/2}[(\Omega-1)(\Omega+\Xi+1)]^{1/2} \quad (A3.3)$$

$$([\Xi+1]\langle\Omega+2\rangle|||\gamma(\mathfrak{X})|||[\Xi+1]\langle\Omega+1\rangle)=-(\Xi+1)^{-1/2}[(\Xi+2)\Omega]^{1/2} \quad (A3.4)$$

$$([\Xi]\langle\Omega+2\rangle|||\gamma(\mathfrak{X})|||[\Xi+1]\langle\Omega+1\rangle)$$
$$=-[(\Xi+1)(2\Xi+1)]^{-1/2}[\Xi(2\Xi+3)(\Omega+\Xi+1)]^{1/2} \quad (A3.5)$$

$$([\Xi+1]\langle\Omega+2\rangle|||\gamma(\mathfrak{X})|||[\Xi]\langle\Omega+1\rangle)=[(\Xi+1)(\Omega+1)]^{-1/2}[\Xi\Omega(\Omega-\Xi)]^{1/2} \quad (A3.6)$$

$$([\Xi]\langle\Omega+2\rangle|||\gamma(\mathfrak{X})|||[\Xi]\langle\Omega+1\rangle)$$
$$=-[\Xi(\Xi+1)(\Omega+1)]^{-1/2}[(\Omega-\Xi)(\Omega+\Xi+1)]^{1/2} \quad (A3.7)$$

$$([\Xi-1]\langle\Omega+2\rangle|||\gamma(\mathfrak{X})|||[\Xi]\langle\Omega+1\rangle)=-[\Xi(\Omega+1)]^{-1/2}[(\Xi+1)\Omega(\Omega+\Xi+1)]^{1/2} \quad (A3.8)$$

$$([\Xi]\langle\Omega+2\rangle|||\gamma(\mathfrak{X})|||[\Xi-1]\langle\Omega+1\rangle)$$
$$=[\Xi(2\Xi+1)]^{-1/2}[(\Xi+1)(2\Xi-1)(\Omega-\Xi)]^{1/2} \quad (A3.9)$$

$$([\Xi-1]\langle\Omega+2\rangle|||\gamma(\mathfrak{X})|||[\Xi-1]\langle\Omega+1\rangle)=\Xi^{-1/2}[(\Xi-1)\Omega]^{1/2} \quad (A3.10)$$

$$([\Xi]\langle\Omega+3\rangle|||\gamma(\mathfrak{X})|||[\Xi+1]\langle\Omega+2\rangle)=-(2\Xi+1)^{-1/2}[(2\Xi+3)(\Omega+\Xi+1)]^{1/2} \quad (A3.11)$$

$$([\Xi]\langle\Omega+3\rangle|||\gamma(\mathfrak{X})|||[\Xi]\langle\Omega+2\rangle)=\Omega^{1/2} \quad (A3.12)$$

$$([\Xi]\langle\Omega+3\rangle|||\gamma(\mathfrak{X})|||[\Xi-1]\langle\Omega+2\rangle)=-(2\Xi+1)^{-1/2}[(2\Xi-1)(\Omega-\Xi)]^{1/2}. \quad (A3.13)$$

The remaining non-vanishing reduced matrix elements $([\xi']\langle\omega-1\rangle|||\gamma(\mathfrak{X})|||[\xi]\langle\omega\rangle)$ can be obtained from them by using the symmetry relation

$$([\xi']\langle\omega-1\rangle|||\gamma(\mathfrak{X})|||[\xi]\langle\omega\rangle)$$
$$=(-1)^{\xi'-\xi}\left(\frac{(2\xi+1)(\omega-1)}{(2\xi'+1)(\omega-2)}\right)^{1/2}([\xi]\langle\omega\rangle|||\gamma(\mathfrak{X})|||[\xi']\langle\omega-1\rangle). \quad (A3.14)$$

Appendix 4. Triple reduced matrix elements of the osp(4/2, ℝ) odd generators

In the generic case corresponding to $\Omega > \Xi_1 > |\Xi_2| + 1$, the so(4)⊕sp(2, ℝ) (triple) reduced matrix elements $(r'[\xi_1'\xi_2']\langle\omega+1\rangle|||\gamma(\mathfrak{T})|||r[\xi_1\xi_2]\langle\omega\rangle)$ of the osp(4/2, ℝ) odd generators are given by

$$([\Xi_1 \pm 1 \Xi_2]\langle\Omega+1\rangle|||\gamma(\mathfrak{T})|||[\Xi_1\Xi_2]\langle\Omega\rangle) = -\Omega^{-1/2}[(\Omega-1)(\Omega \mp \Xi_1 + 1 \mp 1)]^{1/2} \tag{A4.1}$$

$$([\Xi_1\Xi_2 \pm 1]\langle\Omega+1\rangle|||\gamma(\mathfrak{T})|||[\Xi_1\Xi_2]\langle\Omega\rangle) = -\Omega^{-1/2}[(\Omega-1)(\Omega \mp \Xi_2 + 1)]^{1/2} \tag{A4.2}$$

$$\begin{aligned}&([\Xi_1+1\Xi_2 \pm 1]\langle\Omega+2\rangle|||\gamma(\mathfrak{T})|||[\Xi_1+1\Xi_2]\langle\Omega+1\rangle)\\ &\qquad = -[(\Xi_1 \mp \Xi_2 + 1)(\Omega+1)]^{-1/2}[(\Xi_1 \mp \Xi_2 + 2)\Omega(\Omega \mp \Xi_2 + 1)]^{1/2}\end{aligned} \tag{A4.3}$$

$$\begin{aligned}&([\Xi_1+1\Xi_2 \pm 1]\langle\Omega+2\rangle|||\gamma(\mathfrak{T})|||[\Xi_1\Xi_2 \pm 1]\langle\Omega+1\rangle)\\ &\qquad = [(\Xi_1 \mp \Xi_2 + 1)(\Omega+1)]^{-1/2}[(\Xi_1 \mp \Xi_2)\Omega(\Omega - \Xi_1)]^{1/2}\end{aligned} \tag{A4.4}$$

$$\begin{aligned}&([\Xi_1-1\Xi_2 \mp 1]\langle\Omega+2\rangle|||\gamma(\mathfrak{T})|||[\Xi_1\Xi_2 \mp 1]\langle\Omega+1\rangle)\\ &\qquad = -[(\Xi_1 \mp \Xi_2 + 1)(\Omega+1)]^{-1/2}[(\Xi_1 \mp \Xi_2 + 2)\Omega(\Omega + \Xi_1 + 2)]^{1/2}\end{aligned} \tag{A4.5}$$

$$\begin{aligned}&([\Xi_1-1\Xi_2 \mp 1]\langle\Omega+2\rangle|||\gamma(\mathfrak{T})|||[\Xi_1-1\Xi_2]\langle\Omega+1\rangle)\\ &\qquad = [(\Xi_1 \mp \Xi_2 + 1)(\Omega+1)]^{-1/2}[(\Xi_1 \mp \Xi_2)\Omega(\Omega \pm \Xi_2 + 1)]^{1/2}\end{aligned} \tag{A4.6}$$

$$\begin{aligned}&(r[\Xi_1\Xi_2]\langle\Omega+2\rangle|||\gamma(\mathfrak{T})|||[\Xi_1 \pm 1\Xi_2]\langle\Omega+1\rangle)\\ &\qquad = \mp[2(\Xi_1+\Xi_2+1)(\Xi_1-\Xi_2+1)(\Omega+1)(\Omega \mp \Xi_1 + 1 \mp 1)]^{-1/2}\Omega^{1/2}\\ &\qquad\quad \times\{[(\Xi_1 \pm \Xi_2)(\Xi_1 \mp \Xi_2 + 2)]^{1/2}(\mathscr{K}([\Xi_1\Xi_2]\{\Omega+2\}))_{r,[1\,1]}\\ &\qquad\quad + [(\Xi_1 \mp \Xi_2)(\Xi_1 \pm \Xi_2 + 2)]^{1/2}(\mathscr{K}([\Xi_1\Xi_2]\{\Omega+2\}))_{r,[1\,-1]}\}\end{aligned} \tag{A4.7}$$

$$\begin{aligned}&(r[\Xi_1\Xi_2]\langle\Omega+2\rangle|||\gamma(\mathfrak{T})|||[\Xi_1\Xi_2 \pm 1]\langle\Omega+1\rangle)\\ &\qquad = [2(\Xi_1+\Xi_2+1)(\Xi_1-\Xi_2+1)(\Omega+1)(\Omega \mp \Xi_2 + 1)]^{-1/2}\Omega^{1/2}\\ &\qquad\quad \times\{[(\Xi_1+\Xi_2)(\Xi_1-\Xi_2)]^{1/2}(\mathscr{K}([\Xi_1\Xi_2]\{\Omega+2\}))_{r,[1\,\pm 1]}\\ &\qquad\quad - [(\Xi_1+\Xi_2+2)(\Xi_1-\Xi_2+2)]^{1/2}(\mathscr{K}([\Xi_1\Xi_2]\{\Omega+2\}))_{r,[1\,\mp 1]}\}\end{aligned} \tag{A4.8}$$

$$\begin{aligned}&([\Xi_1+1\Xi_2]\langle\Omega+3\rangle|||\gamma(\mathfrak{T})|||[\Xi_1+1\Xi_2 \pm 1]\langle\Omega+2\rangle)\\ &\qquad = \mp(\Xi_1 \pm \Xi_2 + 2)^{-1/2}[(\Xi_1 \pm \Xi_2 + 3)(\Omega \pm \Xi_2 + 1)]^{1/2}\end{aligned} \tag{A4.9}$$

$$\begin{aligned}&([\Xi_1\Xi_2 \pm 1]\langle\Omega+3\rangle|||\gamma(\mathfrak{T})|||[\Xi_1+1\Xi_2 \pm 1]\langle\Omega+2\rangle)\\ &\qquad = \mp(\Xi_1 \pm \Xi_2 + 2)^{-1/2}[(\Xi_1 \pm \Xi_2 + 3)(\Omega + \Xi_1 + 2)]^{1/2}\end{aligned} \tag{A4.10}$$

$$\begin{aligned}&([\Xi_1\Xi_2 \mp 1]\langle\Omega+3\rangle|||\gamma(\mathfrak{T})|||[\Xi_1-1\Xi_2 \mp 1]\langle\Omega+2\rangle)\\ &\qquad = \pm(\Xi_1 \pm \Xi_2)^{-1/2}[(\Xi_1 \pm \Xi_2 - 1)(\Omega - \Xi_1)]^{1/2}\end{aligned} \tag{A4.11}$$

$$\begin{aligned}&([\Xi_1-1\Xi_2]\langle\Omega+3\rangle|||\gamma(\mathfrak{T})|||[\Xi_1-1\Xi_2 \mp 1]\langle\Omega+2\rangle)\\ &\qquad = \pm(\Xi_1 \pm \Xi_2)^{-1/2}[(\Xi_1 \pm \Xi_2 - 1)(\Omega \mp \Xi_2 + 1)]^{1/2}\end{aligned} \tag{A4.12}$$

$$\begin{aligned}&([\Xi_1 \pm 1\Xi_2]\langle\Omega+3\rangle|||\gamma(\mathfrak{T})|||r[\Xi_1\Xi_2]\langle\Omega+2\rangle)\\ &\qquad = \pm[2(\Xi_1+\Xi_2+1\pm 1)(\Xi_1-\Xi_2+1\pm 1)]^{-1/2}\\ &\qquad\quad \times[(\Omega \mp \Xi_1 + 1 \mp 1)(\Omega+\Xi_2+1)(\Omega-\Xi_2+1)]^{1/2}\\ &\qquad\quad \times\{[(\Xi_1 \pm \Xi_2)(\Xi_1 \mp \Xi_2 + 2)]^{1/2}(\mathscr{K}^{-1}([\Xi_1\Xi_2]\{\Omega+2\}))_{[1\,1],r}\\ &\qquad\quad - [(\Xi_1 \mp \Xi_2)(\Xi_1 \pm \Xi_2 + 2)]^{1/2}(\mathscr{K}^{-1}([\Xi_1\Xi_2]\{\Omega+2\}))_{[1\,-1],r}\}\end{aligned} \tag{A4.13}$$

$$([\Xi_1\Xi_2\pm 1]\langle\Omega+3\rangle|||\gamma(\mathfrak{T})|||r[\Xi_1\Xi_2]\langle\Omega+2\rangle)$$
$$=\pm[2(\Xi_1\mp\Xi_2)(\Xi_1\pm\Xi_2+2)]^{-1/2}[(\Omega-\Xi_1)(\Omega+\Xi_1+2)(\Omega\mp\Xi_2+1)]^{1/2}$$
$$\times\{[(\Xi_1+\Xi_2)(\Xi_1-\Xi_2)]^{1/2}(\mathcal{K}^{-1}([\Xi_1\Xi_2]\{\Omega+2\}))_{[1\ \pm1],r}$$
$$+[(\Xi_1+\Xi_2+2)(\Xi_1-\Xi_2+2)]^{1/2}(\mathcal{K}^{-1}([\Xi_1\Xi_2]\{\Omega+2\}))_{[1\ \mp1],r}\} \qquad (A4.14)$$

$$([\Xi_1\Xi_2]\langle\Omega+4\rangle|||\gamma(\mathfrak{T})|||[\Xi_1\pm 1\Xi_2]\langle\Omega+3\rangle)$$
$$=-[(\Xi_1+\Xi_2+1)(\Xi_1-\Xi_2+1)]^{-1/2}$$
$$\times[(\Xi_1+\Xi_2+1\pm 1)(\Xi_1-\Xi_2+1\pm 1)(\Omega\pm\Xi_1+1\pm 1)]^{1/2} \qquad (A4.15)$$

$$([\Xi_1\Xi_2]\langle\Omega+4\rangle|||\gamma(\mathfrak{T})|||[\Xi_1\Xi_2\pm 1]\langle\Omega+3\rangle)$$
$$=[(\Xi_1+\Xi_2+1)(\Xi_1-\Xi_2+1)]^{-1/2}$$
$$\times[(\Xi_1\mp\Xi_2)(\Xi_1\pm\Xi_2+2)(\Omega\pm\Xi_2+1)]^{1/2}. \qquad (A4.16)$$

In (A4.7), (A4.8), (A4.13) and (A4.14), the upper (lower) signs correspond to $r=1$ ($r=2$), and the matrix elements of $\mathcal{K}([\Xi_1\Xi_2]\{\Omega+2\})$ and $\mathcal{K}^{-1}([\Xi_1\Xi_2]\{\Omega+2\})$ result from (I5.16) and (I5.17) when one takes (4.17)-(4.20) into account.

The remaining non-vanishing reduced matrix elements

$$(r'[\xi_1'\xi_2']\langle\omega-1\rangle|||\gamma(\mathfrak{T})|||r[\xi_1\xi_2]\langle\omega\rangle)$$

can be obtained from those listed above by using the symmetry relation

$$(r'[\xi_1'\xi_2']\langle\omega-1\rangle|||\gamma(\mathfrak{T})|||r[\xi_1\xi_2]\langle\omega\rangle)$$
$$=(-1)^{\xi_1'-\xi_1}\left(\frac{(\xi_1+\xi_2+1)(\xi_1-\xi_2+1)(\omega-1)}{(\xi_1'+\xi_2'+1)(\xi_1'-\xi_2'+1)(\omega-2)}\right)^{1/2}$$
$$\times(r[\xi_1\xi_2]\langle\omega\rangle|||\gamma(\mathfrak{T})|||r'[\xi_1'\xi_2']\langle\omega-1\rangle). \qquad (A4.17)$$

In those cases where one of the conditions $\Omega>\Xi_1$, $\Xi_1>|\Xi_2|+1$, or both, are not fulfilled, equations (A4.1)-(A4.6), (A4.9)-(A4.12), (A4.15), and (A4.16) remain valid provided the matrix elements corresponding to forbidden states are left out. The same is true for (A4.7), (A4.8), (A4.13) and (A4.14) whenever $\Omega>\Xi_1=|\Xi_2|+1$.

Whenever $\Omega=\Xi_1>|\Xi_2|$, equation (A4.14) disappears while equations (A4.7), (A4.8) and (A4.13) are replaced by

$$(1[\Xi_1\Xi_2]\langle\Xi_1+2\rangle|||\gamma(\mathfrak{T})|||[\Xi_1-1\Xi_2]\langle\Xi_1+1\rangle)$$
$$=\Xi_2\sqrt{2}\,[(\Xi_1+1)(\Xi_1+\Xi_2+1)(\Xi_1-\Xi_2+1)]^{-1/2} \qquad (A4.18)$$

$$(1[\Xi_1\Xi_2]\langle\Xi_1+2\rangle|||\gamma(\mathfrak{T})|||[\Xi_1\Xi_2\pm 1]\langle\Xi_1+1\rangle)$$
$$=\mp(\Xi_1\pm\Xi_2+1)^{-1/2}[(\Xi_1\pm\Xi_2)(\Xi_1\pm\Xi_2+2)]^{1/2} \qquad (A4.19)$$

$$([\Xi_1-1\Xi_2]\langle\Xi_1+3\rangle|||\gamma(\mathfrak{T})|||1[\Xi_1\Xi_2]\langle\Xi_1+2\rangle)$$
$$=[(\Xi_1+\Xi_2)(\Xi_1-\Xi_2)]^{-1/2}[2\Xi_1(\Xi_1+\Xi_2+1)(\Xi_1-\Xi_2+1)]^{1/2}. \qquad (A4.20)$$

Whenever $\Omega>\Xi_1=|\Xi_2|$, equations (A4.7), (A4.8), (A4.13) and (A4.14), respectively, become

$$([\Xi_1\pm\Xi_1]\langle\Omega+2\rangle|||\gamma(\mathfrak{T})|||[\Xi_1+1\pm\Xi_1]\langle\Omega+1\rangle)$$
$$=-[(2\Xi_1+1)(\Omega+1)]^{-1/2}[2\Xi_1(\Omega+2)(\Omega+\Xi_1+1)]^{1/2} \qquad (A4.21)$$

$$([\Xi_1 \pm \Xi_1]\langle\Omega+2\rangle|||\gamma(\mathfrak{T})|||[\Xi_1 \pm \Xi_1 \mp 1]\langle\Omega+1\rangle)$$
$$= -[(2\Xi_1+1)(\Omega+1)]^{-1/2}[2(\Xi_1+1)(\Omega+2)(\Omega-\Xi_1)]^{1/2} \tag{A4.22}$$

$$([\Xi_1+1 \pm \Xi_1]\langle\Omega+3\rangle|||\gamma(\mathfrak{T})|||[\Xi_1 \pm \Xi_1]\langle\Omega+2\rangle)$$
$$= \pm[2(\Xi_1+1)(\Omega+2)]^{-1/2}[\Xi_1\Omega(\Omega-\Xi_1+1)]^{1/2} \tag{A4.23}$$

$$([\Xi_1 \pm \Xi_1 \mp 1]\langle\Omega+3\rangle|||\gamma(\mathfrak{T})|||[\Xi_1 \pm \Xi_1]\langle\Omega+2\rangle)$$
$$= \mp[2\Xi_1(\Omega+2)]^{-1/2}[(\Xi_1+1)\Omega(\Omega+\Xi_1+2)]^{1/2}. \tag{A4.24}$$

Finally, whenever $\Omega = \Xi_1 = |\Xi_2|$, the four equations have to be left out.

Appendix 5. Reduced matrix elements of the osp(2/2N, ℝ) odd generators for the most degenerate irreps

In the generic case corresponding to $\Omega - N + 1 > |\Xi|$, the u(N) reduced matrix elements of the osp(2/2N, ℝ) odd raising generators between two lowest-weight so(2)⊕u(N) irrep basis states of the $[\Xi\dot{\Omega}\rangle$ star irrep carrier space are given by

$$([\xi \pm 1]\langle\omega+\Delta^{(1)}(k+1)\rangle\{\dot{0}\}\{\omega+\Delta^{(1)}(k+1)\}\|\gamma(\boldsymbol{X})\|[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})$$
$$= (-1)^{(l-\lambda-k\pm k)/2}\tfrac{1}{2}[(2\Omega-l+1)(l-2k)]^{-1/2}$$
$$\times[(k+1)(2\Omega-k+2)(l\mp\lambda-2k)(2\Omega\mp 2\Xi-l\mp\lambda)]^{1/2} \tag{A5.1}$$

$$([\xi \pm 1]\langle\omega+\Delta^{(1)}(l-k+1)\rangle\{\dot{0}\}\{\omega+\Delta^{(1)}(l-k+1)\}\|\gamma(\boldsymbol{X})\|[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})$$
$$= (-1)^{(k\pm k)/2}\tfrac{1}{2}[(2\Omega-l+1)(l-2k+2)]^{-1/2}$$
$$\times[(l-k+2)(2\Omega-l+k+1)(l\pm\lambda-2k+2)(2\Omega\mp 2\Xi-l\mp\lambda)]^{1/2} \tag{A5.2}$$

where $\boldsymbol{X} = \boldsymbol{I}$ (respectively, $\boldsymbol{H}$) for the upper (respectively, lower) signs, $[\xi] = [\Xi+\lambda]$, and $\langle\omega\rangle = \langle(\Omega+2)^k(\Omega+1)^{l-2k}\dot{\Omega}\rangle$.

The reduced matrix elements of the odd lowering generators can be obtained from those of the raising generators by using the symmetry relation

$$([\xi-1]\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\gamma(\boldsymbol{G})\|[\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\})$$
$$= (-1)^{\varphi(\{\omega'\})+\varphi(\{1\dot{0}\})-\varphi(\{\omega\})}\left(\frac{\dim\{\omega\}}{\dim\{\omega'\}}\right)^{1/2}$$
$$\times([\xi]\langle\omega\rangle\{\dot{0}\}\{\omega\}\|\gamma(\boldsymbol{I})\|[\xi-1]\langle\omega'\rangle\{\dot{0}\}\{\omega'\}) \tag{A5.3}$$

and a similar relation connecting $\gamma(\boldsymbol{J})$ with $\gamma(\boldsymbol{H})$. The phase factor $\varphi(\{\omega\})$ has been defined in (I5.26).

For osp(2/2, ℝ), the sp(2, ℝ) (triple) reduced matrix elements of the odd generators $\boldsymbol{J} = (I, J)$ are given by

$$([\Xi+1]\langle\Omega+1\rangle|||\gamma(\boldsymbol{\mathcal{J}})|||[\Xi]\langle\Omega\rangle) = -\Omega^{-1/2}[(\Omega-1)(\Omega-\Xi)]^{1/2} \tag{A5.4}$$
$$([\Xi]\langle\Omega+2\rangle|||\gamma(\boldsymbol{\mathcal{J}})|||[\Xi-1]\langle\Omega+1\rangle) = (\Omega-\Xi)^{1/2} \tag{A5.5}$$
$$([\Xi]\langle\Omega\rangle|||\gamma(\boldsymbol{\mathcal{J}})|||[\Xi-1]\langle\Omega+1\rangle) = (\Omega+\Xi)^{1/2} \tag{A5.6}$$
$$([\Xi+1]\langle\Omega+1\rangle|||\gamma(\boldsymbol{\mathcal{J}})|||[\Xi]\langle\Omega+2\rangle) = \Omega^{-1/2}[(\Omega+1)(\Omega+\Xi)]^{1/2}. \tag{A5.7}$$

Those of $\mathscr{H}=(H, G)$ can be obtained from them by using the symmetry relation

$$([\xi-1]\langle\omega'\rangle\|\gamma(\mathscr{H})\|\|[\xi]\langle\omega\rangle)=-\left(\frac{\omega-1}{\omega'-1}\right)^{1/2}([\xi]\langle\omega\rangle\|\gamma(\mathscr{I})\|\|[\xi-1]\langle\omega'\rangle) \tag{A5.8}$$

resulting from the property $\mathscr{H}_q=(\mathscr{I}_{-q})^{+}$, $q=\frac{1}{2},-\frac{1}{2}$.

Appendix 6. Reduced matrix elements of the osp(2/4, ℝ) odd generators

In the generic case corresponding to $\Omega_1>\Omega_2+1>|\Xi|+2$, the u(2) reduced matrix elements of the osp(2/4, ℝ) odd raising generators between two lowest-weight so(2)⊕ u(2) irrep basis states of the $[\Xi\Omega_1\Omega_2\rangle$ star irrep carrier space are given by

$$([\Xi\pm1]\langle\Omega_1+1\Omega_2\rangle\{\dot{0}\}\{\Omega_1+1\Omega_2\}\|\gamma(\boldsymbol{X})\|[\Xi]\langle\Omega_1\Omega_2\rangle\{\dot{0}\}\{\Omega_1\Omega_2\})=(\Omega_1\mp\Xi)^{1/2} \tag{A6.1}$$

$$([\Xi\pm1]\langle\Omega_1\Omega_2+1\rangle\{\dot{0}\}\{\Omega_1\Omega_2+1\}\|\gamma(\boldsymbol{X})\|[\Xi]\langle\Omega_1\Omega_2\rangle\{\dot{0}\}\{\Omega_1\Omega_2\})=(\Omega_2\mp\Xi-1)^{1/2} \tag{A6.2}$$

$$\begin{aligned}([\Xi\pm2]\langle\Omega_1+1\Omega_2+1\rangle\{\dot{0}\}\{\Omega_1+1\Omega_2+1\}\|\gamma(\boldsymbol{X})\|[\Xi\pm1]\langle\Omega_1+1\Omega_2\rangle\{\dot{0}\}\{\Omega_1+1\Omega_2\})\\ =(\Omega_1-\Omega_2+1)^{-1/2}[(\Omega_1-\Omega_2+2)(\Omega_2\mp\Xi-1)]^{1/2}\end{aligned} \tag{A6.3}$$

$$\begin{aligned}([\Xi\pm2]\langle\Omega_1+1\Omega_2+1\rangle\{\dot{0}\}\{\Omega_1+1\Omega_2+1\}\|\gamma(\boldsymbol{X})\|[\Xi\pm1]\langle\Omega_1\Omega_2+1\rangle\{\dot{0}\}\{\Omega_1\Omega_2+1\})\\ =-(\Omega_1-\Omega_2+1)^{-1/2}[(\Omega_1-\Omega_2)(\Omega_1\mp\Xi)]^{1/2}\end{aligned} \tag{A6.4}$$

$$\begin{aligned}([\Xi]\langle\Omega_1+2\Omega_2\rangle\{\dot{0}\}\{\Omega_1+2\Omega_2\}\|\gamma(\boldsymbol{X})\|[\Xi\mp1]\langle\Omega_1+1\Omega_2\rangle\{\dot{0}\}\{\Omega_1+1\Omega_2\})\\ =\mp\Omega_1^{-1/2}[(\Omega_1+1)(\Omega_1\mp\Xi)]^{1/2}\end{aligned} \tag{A6.5}$$

$$\begin{aligned}(r[\Xi]\langle\Omega_1+1\Omega_2+1\rangle\{\dot{0}\}\{\Omega_1+1\Omega_2+1\}\|\gamma(\boldsymbol{X})\|[\Xi\mp1]\langle\Omega_1+1\Omega_2\rangle\{\dot{0}\}\{\Omega_1+1\Omega_2\})\\ =[2(\Omega_1-\Omega_2+1)(\Omega_1\pm\Xi)]^{-1/2}\{\mp(\Omega_1-\Omega_2)^{1/2}(\mathscr{K}([\Xi]\{\Omega_1+1\Omega_2+1\})_{r,\{20\}}\\ +(\Omega_1-\Omega_2+2)^{1/2}(\mathscr{K}([\Xi]\{\Omega_1+1\Omega_2+1\}))_{r,\{11\}}\}\end{aligned} \tag{A6.6}$$

$$\begin{aligned}(r[\Xi]\langle\Omega_1+1\Omega_2+1\rangle\{\dot{0}\}\{\Omega_1+1\Omega_2+1\}\|\gamma(\boldsymbol{X})\|[\Xi\mp1]\langle\Omega_1\Omega_2+1\rangle\{\dot{0}\}\{\Omega_1\Omega_2+1\})\\ =[2(\Omega_1-\Omega_2+1)(\Omega_2\pm\Xi-1)]^{-1/2}\\ \times\{\mp(\Omega_1-\Omega_2+2)^{1/2}(\mathscr{K}([\Xi]\{\Omega_1+1\Omega_2+1\}))_{r,\{20\}}\\ -(\Omega_1-\Omega_2)^{1/2}(\mathscr{K}([\Xi]\{\Omega_1+1\Omega_2+1\}))_{r,\{11\}}\}\end{aligned} \tag{A6.7}$$

$$\begin{aligned}([\Xi]\langle\Omega_1\Omega_2+2\rangle\{\dot{0}\}\{\Omega_1\Omega_2+2\}\|\gamma(\boldsymbol{X})\|[\Xi\mp1]\langle\Omega_1\Omega_2+1\rangle\{\dot{0}\}\{\Omega_1\Omega_2+1\})\\ =\mp(\Omega_2-1)^{-1/2}[\Omega_2(\Omega_2\mp\Xi-1)]^{1/2}\end{aligned} \tag{A6.8}$$

$$\begin{aligned}([\Xi\pm1]\langle\Omega_1+2\Omega_2+1\rangle\{\dot{0}\}\{\Omega_1+2\Omega_2+1\}\|\gamma(\boldsymbol{X})\|[\Xi]\langle\Omega_1+2\Omega_2\rangle\{\dot{0}\}\{\Omega_1+2\Omega_2\})\\ =\mp[(\Omega_1-\Omega_2+2)(\Omega_1+\Omega_2-1)]^{-1/2}\\ \times[(\Omega_1-\Omega_2+3)(\Omega_1+\Omega_2)(\Omega_2\mp\Xi-1)]^{1/2}\end{aligned} \tag{A6.9}$$

$$\begin{aligned}([\Xi\pm1]\langle\Omega_1+2\Omega_2+1\rangle\{\dot{0}\}\{\Omega_1+2\Omega_2+1\}\|\gamma(\boldsymbol{X})\|r[\Xi]\langle\Omega_1+1\Omega_2+1\rangle\{\dot{0}\}\{\Omega_1+1\Omega_2+1\})\\ =[2\Omega_1(\Omega_1-\Omega_2+2)(\Omega_1+\Omega_2-1)]^{-1/2}\\ \times[(\Omega_1+1)(\Omega_1+\Omega_2)(\Omega_1+\Xi)(\Omega_1-\Xi)(\Omega_2\mp\Xi-1)]^{1/2}\\ \times\{\pm(\Omega_1-\Omega_2)^{1/2}(\mathscr{K}^{-1}([\Xi]\{\Omega_1+1\Omega_2+1\}))_{\{20\},r}\\ -(\Omega_1-\Omega_2+2)^{1/2}(\mathscr{K}^{-1}([\Xi]\{\Omega_1+1\Omega_2+1\}))_{\{11\},r}\}\end{aligned} \tag{A6.10}$$

$$([\Xi\pm 1]\langle\Omega_1+1\Omega_2+2\rangle\{\dot 0\}\{\Omega_1+1\Omega_2+2\}\|\gamma(\boldsymbol{X})\|r[\Xi]\langle\Omega_1+1\Omega_2+1\rangle\{\dot 0\}\{\Omega_1+1\Omega_2+1\})$$
$$=[2(\Omega_2-1)(\Omega_1-\Omega_2)(\Omega_1+\Omega_2-1)]^{-1/2}$$
$$\times[\Omega_2(\Omega_1+\Omega_2)(\Omega_1\mp\Xi)(\Omega_2+\Xi-1)(\Omega_2-\Xi-1)]^{1/2}$$
$$\times\{\mp(\Omega_1-\Omega_2+2)^{1/2}(\mathcal{K}^{-1}([\Xi]\{\Omega_1+1\Omega_2+1\}))_{\{20\},r}$$
$$-(\Omega_1-\Omega_2)^{1/2}(\mathcal{K}^{-1}([\Xi]\{\Omega_1+1\Omega_2+1\}))_{\{11\},r}\} \tag{A6.11}$$

$$([\Xi\pm 1]\langle\Omega_1+1\Omega_2+2\rangle\{\dot 0\}\{\Omega_1+1\Omega_2+2\}\|\gamma(\boldsymbol{X})\|[\Xi]\langle\Omega_1\Omega_2+2\rangle\{\dot 0\}\{\Omega_1\Omega_2+2\})$$
$$=\pm[(\Omega_1-\Omega_2)(\Omega_1+\Omega_2-1)]^{-1/2}$$
$$\times[(\Omega_1-\Omega_2-1)(\Omega_1+\Omega_2)(\Omega_1\mp\Xi)]^{1/2} \tag{A6.12}$$

$$([\Xi\mp 1]\langle\Omega_1+2\Omega_2+1\rangle\{\dot 0\}\{\Omega_1+2\Omega_2+1\}\|\gamma(\boldsymbol{X})\|[\Xi\mp 2]\langle\Omega_1+1\Omega_2+1\rangle\{\dot 0\}\{\Omega_1+1\Omega_2+1\})$$
$$=[\Omega_1(\Omega_1+\Omega_2-1)]^{-1/2}[(\Omega_1+1)(\Omega_1+\Omega_2)(\Omega_1\mp\Xi)]^{1/2} \tag{A6.13}$$

$$([\Xi\mp 1]\langle\Omega_1+1\Omega_2+2\rangle\{\dot 0\}\{\Omega_1+1\Omega_2+2\}\|\gamma(\boldsymbol{X})\|[\Xi\mp 2]\langle\Omega_1+1\Omega_2+1\rangle\{\dot 0\}\{\Omega_1+1\Omega_2+1\})$$
$$=[(\Omega_2-1)(\Omega_1+\Omega_2-1)]^{-1/2}[\Omega_2(\Omega_1+\Omega_2)(\Omega_2\mp\Xi-1)]^{1/2} \tag{A6.14}$$

$$([\Xi]\langle\Omega_1+2\Omega_2+2\rangle\{\dot 0\}\{\Omega_1+2\Omega_2+2\}\|\gamma(\boldsymbol{X})\|[\Xi\mp 1]\langle\Omega_1+2\Omega_2+1\rangle\{\dot 0\}\{\Omega_1+2\Omega_2+1\})$$
$$=[(\Omega_2-1)(\Omega_1-\Omega_2+1)(\Omega_1+\Omega_2)]^{-1/2}$$
$$\times[\Omega_2(\Omega_1-\Omega_2+2)(\Omega_1+\Omega_2+1)(\Omega_2\mp\Xi-1)]^{1/2} \tag{A6.15}$$

$$([\Xi]\langle\Omega_1+2\Omega_2+2\rangle\{\dot 0\}\{\Omega_1+2\Omega_2+2\}\|\gamma(\boldsymbol{X})\|[\Xi\mp 1]\langle\Omega_1+1\Omega_2+2\rangle\{\dot 0\}\{\Omega_1+1\Omega_2+2\})$$
$$=-[\Omega_1(\Omega_1-\Omega_2+1)(\Omega_1+\Omega_2)]^{-1/2}$$
$$\times[(\Omega_1+1)(\Omega_1-\Omega_2)(\Omega_1+\Omega_2+1)(\Omega_1\mp\Xi)]^{1/2} \tag{A6.16}$$

where $\boldsymbol{X}=\boldsymbol{I}$ (respectively, $\boldsymbol{H}$) for the upper (respectively, lower) signs.

In those cases where one of the conditions $\Omega_1>\Omega_2+1$, $\Omega_2-1>|\Xi|$, or both, are not fulfilled, equations (A6.1)-(A6.5), (A6.8), (A6.9) and (A6.12)-(A6.16) remain valid provided the matrix elements corresponding to forbidden states are left out.

Whenever $\Omega_1>\Omega_2$ and $\Omega_2-1=|\Xi|$, equation (A6.11) disappears while equations (A6.6), (A6.7) and (A6.10) are replaced by

$$(1[\pm\Omega_2\mp 1]\langle\Omega_1+1\Omega_2+1\rangle\{\dot 0\}\{\Omega_1+1\Omega_2+1\}\|\gamma(\boldsymbol{X})\|[\pm\Omega_2\mp 2]\langle\Omega_1+1\Omega_2\rangle\{\dot 0\}\{\Omega_1+1\Omega_2\})$$
$$=[(\Omega_1+\Omega_2-2)(\Omega_1-\Omega_2+1)]^{-1/2}[2(\Omega_2-1)]^{1/2} \tag{A6.17}$$

$$(1[\mp\Omega_2\pm 1]\langle\Omega_1+1\Omega_2+1\rangle\{\dot 0\}\{\Omega_1+1\Omega_2+1\}\|\gamma(\boldsymbol{X})\|[\mp\Omega_2]\langle\Omega_1+1\Omega_2\rangle\{\dot 0\}\{\Omega_1+1\Omega_2\})$$
$$=(\Omega_1+\Omega_2-2)^{-1/2}[2(\Omega_2-1)(\Omega_1+\Omega_2-1)]^{1/2} \tag{A6.18}$$

$$(1[\pm\Omega_2\mp 1]\langle\Omega_1+1\Omega_2+1\rangle\{\dot 0\}\{\Omega_1+1\Omega_2+1\}\|\gamma(\boldsymbol{X})\|[\pm\Omega_2\mp 2]\langle\Omega_1\Omega_2+1\rangle\{\dot 0\}\{\Omega_1\Omega_2+1\})$$
$$=-[(\Omega_1+\Omega_2-2)(\Omega_1-\Omega_2+1)]^{-1/2}$$
$$\times[(\Omega_1+\Omega_2-1)(\Omega_1-\Omega_2)(\Omega_1-\Omega_2+2)]^{1/2} \tag{A6.19}$$

$$([\mp\Omega_2\pm 2]\langle\Omega_1+2\Omega_2+1\rangle\{\dot 0\}\{\Omega_1+2\Omega_2+1\}\|\gamma(\boldsymbol{X})$$
$$\times\|1[\mp\Omega_2\pm 1]\langle\Omega_1+1\Omega_2+1\rangle\{\dot 0\}\{\Omega_1+1\Omega_2+1\})$$
$$=-[\Omega_1(\Omega_1+\Omega_2-1)(\Omega_1-\Omega_2+2)]^{-1/2}$$
$$\times[(\Omega_1+1)(\Omega_1+\Omega_2)(\Omega_1+\Omega_2-2)(\Omega_1-\Omega_2+1)]^{1/2}. \tag{A6.20}$$

In all the cases, the reduced matrix elements of the odd lowering generators can be obtained from those of the odd raising ones by using relations similar to (A5.3).

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